Optimal Sensor Placement for Intruder Detection

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I. INTRODUCTION

The fields of detection, resource allocation, and security have seen tremendous research activity in the past few decades. Different frameworks have been designed to tackle various problems using a host of techniques from statistics, engineering, and economics. The fields of centralized and decentralized detection, see [1] for a review, have been a major research focus in the communications, controls, and statistics communities and are now considered mature fields of research. Our main focus in this paper is to develop and pursue research ideas in the intersection of these exciting fields. In this paper, we operate within the centralized detection framework under which complete observations are available to the decision makers. This preference for a relatively simple and more developed framework enables us to fully extract the benefits offered by the diversity of these research ideas.

We consider the centralized detection of an intruder, whose location is modeled as uniform across a specified set of points. A team of sensors is tasked to make observations which are completely reported to a centralized decision making authority. The challenge is to optimally place the sensors, before measurements are made, and to compute an optimal decision rule for detecting the location of the intruder. Simplifying assumptions like a uniform prior distribution and conditional independence of observations are made. It should be noted that sensors observations are noisy as a result of their limited accuracy. The limited accuracy of the sensors makes the conditional independence assumption practically reasonable [2].

We exploit majorization theory, see [3] and [4] for a review, within the framework of this paper to establish general laws governing the jointly optimal detection and placement policies. Majorization is an important mathematical technique for partial ordering of vectors of real

numbers. It is widely used in mathematical statistics and has recently been applied to solve challenging problems in controls [5] and communications [6].

The following notation is adopted:

- Vectors are represented as $\bar{x} = (x_1, ..., x_k)$, where the vector length would be obvious from the context.

- The observation vector received by the fusion center is denoted by $\bar{y} = (y_1, ..., y_M)$, the position vector of the sensors is denoted by $\bar{u} = (u_1, ..., u_M)$, and the placement vector for the sensors is denoted by $\bar{v} = (v_1, ..., v_N)$.

- Random variables are represented by bold face capital letters, such as $\mathbf{X}$ used to represent the position of the intruder. Realizations are represented using small letters.

- The probability of error is denoted by P_e, the prior distribution by π, the probability density of a random variable $\mathbf{A}$ if it exists by $p(a)$, the joint probability density of $\mathbf{A}$ and $\mathbf{B}$ by $p(a, b)$, and the conditional probability density of $\mathbf{A}$ given $\mathbf{B}$ by $p_b(a)$.

- The observation space is denoted by Ω.

- Capital letters M and N are used to denote the number of sensors and the number of placement points respectively.

The paper is organized into seven main sections. In Section II we provide the technical preliminaries and in Section III we provide the problem formulation. The main results of this paper are presented in Section IV and Section V. In Section IV, we use mathematical induction to establish general principles regarding the strict optimality of the uniform placement. In Section V, majorization theory is exploited to formalize and conjecture important properties of the optimal solution. Simulations and conclusions are presented in Section VI and Section VII.

II. TECHNICAL PRELIMINARIES

In this section, we present definitions of the main concepts to be utilized in the sequel.

Definition 2.1: (**Partition Function**) The partition function $f(N)$ of a positive integer N is defined as the number of ways in which N can be written as a sum of positive integers. For example 4 can be written as $4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1$ and $f(4) = 5$.

Definition 2.2: (**Majorization** [4]) For $\bar{x}, \bar{y} \in \Re^N$, $\bar{x}$ is said to be majorized by $\bar{y}$, $\bar{y} \succ \bar{x}$, provided that: $\sum_{j=1}^{n} x_{[j]} = \sum_{j=1}^{n} y_{[j]}, \sum_{j=1}^{k} x_{[j]} \leq \sum_{j=1}^{k} y_{[j]}, k = 1, ..., n - 1$ where $x_{[j]}, j = 1, ..., n$

represents the elements of $\bar{x}$ in nonincreasing order, $x_{[1]} \geq, ..., \geq x_{[n]}$.

Definition 2.3: (**Majorization Based Placement Scale**) Sensor placements $\bar{\alpha}, \bar{\beta}$, and $\bar{\gamma}$ can be placed on a majorization based scale $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ if the following majorization ordering exists: $\bar{\alpha} \succ \bar{\beta} \succ \bar{\gamma}$. The placement $\bar{\alpha}$ is said to be at the highest level on this scale followed by $\bar{\beta}$ and $\bar{\gamma}$.

III. PROBLEM FORMULATION

In this section a rigorous problem formulation along with a precise problem statement is presented.

A. Sensor Placement, Data Collection, and Performance Criterion

Consider a team of M identical sensors tasked with detecting the location of an intruder $\mathbf{X}$ which can occur on a set of points $\{1, .., N\}$, with a uniform distribution. It is assumed that $M \leq N$. The sensors are placed using a specific sensor position vector (or placement vector), sensors make observations, and report them directly to a fusion center. The observations made by the sensors are assumed to be conditionally independent given the true position of the intruder. The fusion center collects these observations and computes an estimate of the position of the intruder. It should be noted that no quantization of information occurs at the sensors and observations are completely reported to the fusion center. The performance criterion minimized at the fusion center is the probability of error in detecting the actual location of the intruder. The main problem is to design jointly optimal location detection and sensor placement policies.

B. Observation Model

Let $\mathbf{Y}_k$ be the measurement obtained by sensor k, $k = 1, ..., M$. The random variable $\mathbf{Y}_k$ either takes the value 1 (Intruder present) or the value 0 (Intruder not present). The observation model is presented below:

$$Pr(\mathbf{Y}_k = 1 | \mathbf{X} = u_k) = P_D$$

$$Pr(\mathbf{Y}_k = 1 | \mathbf{X} \neq u_k) = P_F$$

where P_D and P_F are the detection and false alarm probabilities of the sensors. We assume that these values have been provided by the manufacturer of these sensors and are well known. It should be noted that the sensors can only make measurements at the point on which they are placed and cannot infer any information about the presence of the intruder at ther points.

C. M-ary Hypothesis Testing

The problem is modeled as an M-ary hypothesis testing problem [7] under the Bayesian formulation. Under a uniform prior distribution, $\pi_i = \frac{1}{N}$, the probability of error at the fusion center for a specific sensor position vector $(u_1, ..., u_M)$ is given by [8]:

$$P_e(u_1, ..., u_M) = \min_{\delta} \frac{1}{N} \sum_{i=1}^{N} \sum_{\bar{y} \in \Gamma_i} \sum_{j=1, j \neq i}^{N} p_j(\bar{y})$$

where $\Gamma_i, i = 1, ..., N$ defines a partition of the observation space Ω such that $\Omega = \bigcup_{i=1}^{N} \Gamma_i$ where $\Gamma_i = \{\bar{y} \in \Omega | \delta(y) = H_i\}$. Here $p_j(\bar{y})$ is the joint probability mass function (pmf) for the random variables $(\mathbf{Y}_1, ..., \mathbf{Y}_M)$ given that $\mathbf{X} = j$. It should be noted that the joint pmf depends on the specific sensor position vector (or placement vector). The sensor position vector, of length M, indicates the points at which each sensor is placed whereas the sensor placement vector, of length N, indicates how many sensors are placed at each point. It should be noted that:

$$1 \leq u_j \leq N, \ j = 1, ..., M \qquad\qquad 0 \leq v_k \leq M, \ k = 1, ..., N$$

Since we consider identical sensors using the same observation model it is sufficient to consider the number of sensors which are placed at each point. The placement vector for the sensors is more convenient to use than the sensor position vector. In the sequel we will utilize the sensor placement vector $\bar{v} = (v_1, ..., v_N)$ instead of the sensor position vector $\bar{u} = (u_1, ..., u_M)$ for the calculation of the probabilities of error.

D. Calculation of Conditional Probabilities

Since the observations are conditionally independent:

$$p_j(\bar{y}) = \prod_{i=1}^{M} p_j(y_i), \ \ j = 1, ..., N$$

We will only consider placements of the form $v_1 \geq v_2 \geq .... \geq v_N$. For the case M=N=5, consider the two sensor placements $(v_1, ..., v_5) = (2, 1, 1, 1, 0)$ and $(v_1, ..., v_5) = (1, 2, 1, 1, 0)$. Since the intruder occurs at any of the five points with a uniform distribution so clearly $P_e(2,1,1,1,0) = P_e(1,2,1,1,0)$. Therefore, we will only consider $(2,1,1,1,0)$ and not other similar permutations of the placement vector in which one point has two sensors, three points have one sensor each, and one point has no sensor placed on it. For notational convenience we assume that during deployment the sensors are placed in an ascending order, i-e first v_1 sensors are deployed on point 1 followed by v_2 sensors on point 2 and, so on. We hasten to add that this deployment policy is really not restrictive; it merely amounts to a relabeling of the observation areas or locations: location 1 is the one with the most number of sensors, and so on. Thus location j need not be 'adjacent' to location $j-1$ or $j+1$. This deployment policy ensures that observations $(y_1, ..., y_{v_1})$ are made by sensors operating at point 1 and similarly we can interpret the observations $(y_{v_1+1}, ..., y_M)$. With $M \leq N$ and the aforementioned assumptions in place, it is clear that $(v_{M+1}, ..., v_N) = (0, ..., 0)$. Note that depending on the values of the v_i's it is possible that $v_k = 0$, for $k = k_0, ..., M$, as well. The conditional probability is calculated as follows:

$$p_j(\bar{y}) = \left\{ (P_F)^{(y_1+...+y_M)} \times (1-P_F)^{M-(y_1+...+y_M)} \chi_{(v_j=0)} \right\} + \left\{ (P_D)^{a_j} (1-P_D)^{v_j-a_j} (P_F)^{(y_1+...+y_M)-a_j} \times \right.$$
$$\left. (1-P_F)^{M-(y_1+...+y_M)-(v_j-a_j)} \chi_{(v_j \neq 0)} \right\} \tag{1}$$

where a_j is the number of accurate 'alarmed' sensors at location j and is given by:

$$a_j = y_1 + ... + y_{v_j}, j = 1$$

$$a_j = y_{(v_1+...+v_{j-1})+1} + ... + y_{(v_1+...+v_{j-1})+v_j}, j = 2, ..., N$$

E. Problem Statement

Given M sensors and N placement points we want to solve the following optimization problem:

$$P_e^* = \min_\delta \min_{v_1, ..., v_M} \frac{1}{N} \sum_{i=1}^{N} \sum_{\bar{y} \in \Gamma_i} \sum_{j=1, j \neq i}^{N} p_j(\bar{y})$$

subject to the constraint:

$$(v_1, ..., v_M) \in \Lambda^M$$

where the set Λ^M is given by an integer partition of M. Utilizing the notion of the partition function defined in Section II, the size of the constraint set Λ^M is given by:

$$|\Lambda^M| = f(M)$$

The partition function $f(M)$ of an integer M can be expressed asymptotically as [9]:

$$f(M) \approx \frac{1}{4M\sqrt{3}} \times e^{\pi\sqrt{\frac{2M}{3}}} \; as \; M \to \infty$$

This expression was first proved by S. Ramanujan and G. Hardy in 1918. The form of the cost function and the exponential complexity of the constraint set make this problem difficult to solve for a general (M,N). We prove important solution properties for this problem in Section IV and Section V. Such properties are the basis of any suboptimal scheme designed to solve this problem. For any placement $(v_1, ..., v_M)$ the probability of error is given by:

$$P_e(v_1, ..., v_M) = \min_\delta \frac{1}{N} \sum_{i=1}^{N} \sum_{\bar{y}\in\Gamma_i} \sum_{j=1,j\neq i}^{N} p_j(\bar{y})$$

Minimization is carried out by optimally selecting $\Gamma_i, i = 1, ..., N$:

$$\Gamma_i^* = \left\{ \bar{y} \in \Omega |\; \sum_{j=1,j\neq i}^{N} p_j(\bar{y}) \leq \sum_{j=1,j\neq k}^{N} p_j(\bar{y}), i \neq k \right\}$$

which can be simplified to give:

$$\Gamma_i^* = \left\{ \bar{y} \in \Omega \mid p_i(\bar{y}) = \max_{k=1,...,N} p_k(\bar{y}) \right\} \tag{2}$$

Using (2) and the uniform distribution of the priors the optimal decision rule is given by:

$$\delta^*(\bar{y}) = \left\{ H_i \mid \pi_{\bar{y}}(H_i) = \max_{k=1,..,N} \pi_{\bar{y}}(H_k) \right\}, \bar{y} \in \Omega \tag{3}$$

where $\pi_{\bar{y}}(H)$ is the posterior distribution. Note that the optimal decision rule in (3) is the maximum a posteriori probability (MAP) rule. Since δ^* is optimal for any arbitrary placement so the optimization problem can be rewritten as follows:

$$P_e^* = \min_{v_1,...,v_M} \frac{1}{N} \sum_{i=1}^{N} \sum_{\bar{y}\in\Gamma_i^*} \sum_{j=1,j\neq i}^{N} p_j(\bar{y}) \tag{4}$$

subject to the constraint:

$$(v_1, ..., v_M) \in \Lambda^M$$

IV. Characterization of the Optimal Solution

We first provide a definition of the strict optimality of sensor placement and then present the results of this section.

Definition 4.1: (**Strict Optimality of Placement**) We call a placement $(v'_1, ..., v'_M)$ strictly optimal on the (P_F, P_D) plane, if there exist P'_D, P'_F such that $P_e(v_1, ..., v_M)|_{P_D=P'_D, P_F=P'_F} > P_e(v'_1, ..., v'_M)|_{P_D=P'_D, P_F=P'_F}$ for any $(v_1, ..., v_M) \in \Lambda^M$.

It should be noted that an optimal placement for (4) is an optimal solution to the aforementioned problem. Using Definition 4.1, the main result of this section is described as follows:

Theorem 4.1: For the case $(M = N)$, the uniform sensor placement $(v_1, ..., v_M) = (1, ..., 1)$ is not strictly optimal for any value of P_D, P_F on the (P_F, P_D) plane.

Proof: We will prove this result by using induction. For notational convenience we will use the following form of $P_e(v_1, ..., v_M)$ in the proof:

$$P_e(v_1, ..., v_M) = \frac{1}{N} \sum_{\bar{y} \in \{0,1\}^M} \min_{i=1,..,N} \{ \sum_{j=1, j \neq i}^{N} p_j(\bar{y}) \} \tag{5}$$

For M sensors we have 2^M observations so $\{0, 1\}^M$ is used to denote the observation space Ω.

First consider the case M=N=2. In this case (5) can be simplified to give:

$$P_e(v_1, v_2) = \frac{1}{2} [\min\{p_1(0, 0), p_2(0, 0)\} + \min\{p_1(0, 1),$$

$$p_2(0, 1)\} + \min\{p_1(1, 0), p_2(1, 0)\} + \min\{p_1(1, 1), p_2(1, 1)\}] \tag{6}$$

Using equation (1) the conditional probabilities for the $(v_1, v_2) = (1, 1)$ and $(v_1, v_2) = (2, 0)$

placements are given as follows:

$$p_1(y_1, y_2)\big|_{(1,1)} = (P_D)^{y_1}(1 - P_D)^{1-y_1}(P_F)^{y_2}(1 - P_F)^{1-y_2}$$

$$p_2(y_1, y_2)\big|_{(1,1)} = (P_D)^{y_2}(1 - P_D)^{1-y_2}(P_F)^{y_1}(1 - P_F)^{1-y_1}$$

$$p_1(y_1, y_2)\big|_{(2,0)} = (P_D)^{y_1+y_2}(1 - P_D)^{2-y_1-y_2}$$

$$p_2(y_1, y_2)\big|_{(2,0)} = (P_F)^{y_1+y_2}(1 - P_F)^{2-y_1-y_2} \tag{7}$$

Plugging (7) into (6) we get:

$$P_e(1,1) = \frac{1}{2}\Big[(1 - P_D)(1 - P_F) + P_D P_F + 2\min\{P_F - P_D P_F, P_D - P_D P_F\}\Big]$$

$$P_e(2,0) = \frac{1}{2}\Big[\min\{(1 - P_D)^2, (1 - P_F)^2\} + \min\{P_D^2, P_F^2\} + 2\min\{P_D(1 - P_D), P_F(1 - P_F)\}\Big] \tag{8}$$

First consider the case $\mathbf{P_D} > \mathbf{P_F}$:

For $P_D > P_F, P_D(1 - P_D) > P_F(1 - P_F)$:

$$P_e(1,1) = \frac{1}{2}\big[1 + P_F - P_D\big]$$

$$P_e(2,0) = \Big[\frac{1}{2} + (P_F - P_D) + \frac{(P_D^2 - P_F^2)}{2}\Big]$$

where $(P_D^2 - P_F^2) - (P_D - P_F) < 0 \Rightarrow P_e(2,0) < P_e(1,1)$

For $P_D > P_F, P_D(1 - P_D) = P_F(1 - P_F)$:

$$P_e(1,1) = P_e(2,0) = \frac{1}{2}\big[1 + P_F - P_D\big]$$

For $P_D > P_F, P_D(1 - P_D) < P_F(1 - P_F)$:

$$P_e(1,1) = \frac{1}{2}\big[1 + (P_F - P_D)\big], P_e(2,0) = \frac{1}{2}\big[1 + (P_F^2 - P_D^2)\big]$$

where $(P_F^2 - P_D^2) < (P_F - P_D) \Rightarrow P_e(2,0) < P_e(1,1)$

For the case $\mathbf{P_D} = \mathbf{P_F}$ we have:

$$P_e(2,0) = P_e(1,1) = \frac{1}{2}$$

For the purposes of this proof it is sufficient to only consider the case $P_D \geq P_F$. Further details regarding this sufficiency are provided at the end of the proof. This proves that uniform placement is not strictly optimal for $M=N=2$.

Assume that the result holds for $M=N=k$:

$$P_e\Big(\underbrace{1,...,1}_{k}\Big) \geq P_e\Big(\underbrace{2,1,...,1,0}_{k}\Big) \tag{9}$$

Calculation of $\mathbf{P_e}\Big(\underbrace{1,...,1}_{k}\Big)$:

Using equations (1) and (5) we get the following expression:

$$P_e\Big(\underbrace{1,...,1}_{k}\Big) = \frac{1}{k}\left[\sum_{\bar{y}\in\{0,1\}^k} \min_{i=1,..,k} \{A_i(\bar{y})\}\right]$$

$$A_i(\bar{y}) = \sum_{j=1,j\neq i}^{k} \left\{ (P_D)^{y_j}(1-P_D)^{1-y_j}(P_F)^{(y_1+...+y_k)-y_j}(1-P_F)^{k-1-(y_1+...+y_k)+y_j}\right\} \tag{10}$$

Calculation of $\mathbf{P_e}\Big(\underbrace{2,1,...,1,0}_{k}\Big)$:

Using equations (1) and (5) we get the following expression:

$$P_e\Big(\underbrace{2,1,...,1,0}_{k}\Big) = \frac{1}{k}\left[\sum_{\bar{y}\in\{0,1\}^k} \min_{i=1,..,k} \{B_i(\bar{y})\}\right] \tag{11}$$

where $B_i(\bar{y}), i = 1, .., k$ is given by:

$$B_1(\bar{y}) = \left[\sum_{j=2}^{k-1} (P_D)^{y_{j+1}} (1 - P_D)^{1-y_{j+1}} (P_F)^{(y_1+...+y_k)-y_{j+1}} (1 - P_F)^{k-1-(y_1+...+y_k)+y_{j+1}} \right]$$

$$+ (P_F)^{(y_1+..+y_k)} (1 - P_F)^{k-(y_1+..+y_k)}$$

$$B_i(\bar{y}) = \left[\sum_{j=2, j\neq i}^{k-1} (P_D)^{y_{j+1}} (1 - P_D)^{1-y_{j+1}} (P_F)^{(y_1+...+y_k)-y_{j+1}} (1 - P_F)^{k-1-(y_1+...+y_k)+y_{j+1}} \right]$$

$$+ (P_F)^{(y_1+..+y_k)} (1 - P_F)^{k-(y_1+..+y_k)} + (P_D)^{y_1+y_2} (1 - P_D)^{2-y_1-y_2} (P_F)^{(y_3+...+y_k)} (1 - P_F)^{k-2-(y_3+...+y_k)}$$

$, i = 2, ..., k - 1$

$$B_k(\bar{y}) = \left[\sum_{j=2}^{k-1} (P_D)^{y_{j+1}} (1 - P_D)^{1-y_{j+1}} (P_F)^{(y_1+...+y_k)-y_{j+1}} (1 - P_F)^{k-1-(y_1+...+y_k)+y_{j+1}} \right] +$$

$$+ (P_D)^{y_1+y_2} (1 - P_D)^{2-y_1-y_2} (P_F)^{(y_3+...+y_k)} (1 - P_F)^{k-2-(y_3+...+y_k)} \tag{12}$$

Using equation (9) we can conclude that:

$$\left[\sum_{\bar{y}\in\{0,1\}^k} \min_{i=1,..,k} \{A_i(\bar{y})\} - \sum_{\bar{y}\in\{0,1\}^k} \min_{i=1,..,k} \{B_i(\bar{y})\} \right] \geq 0$$

In order to prove the Theorem we need to show that:

$$P_e\big(\underbrace{1, ..., 1}_{k+1}\big) \geq P_e\big(\underbrace{2, 1, ..., 1, 0}_{k+1}\big)$$

Calculation of $\mathbf{P_e}\big(\underbrace{1, ..., 1}_{\mathbf{k+1}}\big)$:

Using equations (1), (5) and (10) we get the following expression:

$$P_e\big(\underbrace{1, ..., 1}_{k+1}\big) = \frac{1}{k+1} \left[\sum_{\bar{y}\in\{0,1\}^k} \left\{ \min\left\{ (1 - P_F)A_1(\bar{y}) + c(\bar{y}), ..., (1 - P_F)A_k(\bar{y}) + c(\bar{y}), \right. \right. \right.$$

$$\left. (1 - P_F)A_k(\bar{y}) + c^*(\bar{y}) \right\} + \min\left\{ (P_F)A_1(\bar{y}) + d(\bar{y}), ... (P_F)A_k(\bar{y}) + d(\bar{y}), (P_F)A_k(\bar{y}) + d^*(\bar{y}) \right\} \right\} \bigg] \tag{13}$$

where $c(\bar{y}), c^*(\bar{y}), d(\bar{y}), d^*(\bar{y})$ are provided below:

$$c(\bar{y}) = (1 - P_D)(P_F)^{(y_1 + ... + y_k)}(1 - P_F)^{k-(y_1 + ... + y_k)}$$

$$c^*(\bar{y}) = (P_D)^{y_k}(1 - P_D)^{1-y_k}(P_F)^{(y_1 + ... + y_{k-1})}(1 - P_F)^{k-(y_1 + ... + y_{k-1})}$$

$$d(\bar{y}) = P_D(P_F)^{(y_1 + ... + y_k)}(1 - P_F)^{k-(y_1 + ... + y_k)}$$

$$d^*(\bar{y}) = (P_D)^{y_k}(1 - P_D)^{1-y_k}(P_F)^{y_1 + .. + y_{k-1} + 1}(1 - P_F)^{k-1-(y_1 + .. + y_{k-1})} \tag{14}$$

Calculation of $P_e\big(\underbrace{2, 1, ..., 1, 0}_{k+1}\big)$:

Using equations (1), (5) and (11) we get the following expression:

$$P_e\big(\underbrace{2, 1, ..., 1, 0}_{k+1}\big) = \frac{1}{k+1}\Bigg[\sum_{\bar{y} \in \{0,1\}^k}\Big\{\min\big\{(1 - P_F)B_1(\bar{y}) + c(\bar{y}), ..., (1 - P_F)B_{k-1}(\bar{y}) + c(\bar{y}),$$

$$(1 - P_F)B_{k-1}(\bar{y}) + c^*(\bar{y}), (1 - P_F)B_k(\bar{y}) + c(\bar{y})\big\} + \min\big\{(P_F)B_1(\bar{y}) + d(\bar{y}), ..., (P_F)B_{k-1}(\bar{y}) + d(\bar{y}),$$

$$(P_F)B_{k-1}(\bar{y}) + d^*(\bar{y}), (P_F)B_k(\bar{y}) + d(\bar{y})\big\}\Big\}\Bigg] \tag{15}$$

If $P_D = 1$ or $P_F = 0$ then $c^*(\bar{y}) \geq c(\bar{y})$. If $P_D < 1$ and $P_F > 0$ then $c^*(\bar{y})$ can be expressed in terms of $c(\bar{y})$ as follows:

$$c^*(\bar{y}) = c(\bar{y})\left(\frac{P_D(1 - P_F)}{P_F(1 - P_D)}\right)^{y_k} \tag{16}$$

First consider the case where we have good sensors i-e sensors for which $\mathbf{P_D} \geq \mathbf{P_F}$. From equation (16) we have that $c^*(\bar{y}) \geq c(\bar{y})$. Therefore, we can eliminate the terms involving $c^*(\bar{y})$ from equations (13) and (15). We will first consider the situations where we can eliminate the terms involving $d^*(\bar{y})$ from equation (13) and then proceed by doing a similar analysis for equation (15). If $y_k = 1$, then $d^*(\bar{y}) = d(\bar{y})$. Using equations (10), (14), and $(y_{k-1}, y_k) = (0, 0)$ we obtain:

$$\big(P_F A_k(\bar{y}) + d^*(y)\big) - \big(P_F A_{k-1}(\bar{y}) - d(\bar{y})\big) = (P_F)^{(y_1 + ... + y_{k-2})}(1 - P_F)^{k-1-(y_1 + ... + y_{k-2})}(P_F - P_D) \leq 0 \tag{17}$$

From equations (10), (14), and $(y_{k-1}, y_k) = (1, 0)$ we obtain:

$$(P_F)A_k(\bar{y}) + d^*(\bar{y}) = (P_F)A_{k-1}(\bar{y}) + d(\bar{y}) \tag{18}$$

Similarly it can be shown for $(y_{j-1}, y_k) = (1, 0)$:

$$(P_F)A_k(\bar{y}) + d^*(\bar{y}) = (P_F)A_{j-1}(\bar{y}) + d(\bar{y}), j = 2, ..., k \qquad (19)$$

So for $\bar{y} \in \{0,1\}^k/(0,...,0)$ we can eliminate the term $(P_F)A_k(\bar{y}) + d^*(\bar{y})$ from equation (13). For the observation $\bar{y} = (0,...,0), \min \{(P_F)A_1(\bar{y}) + d(\bar{y}), ..., (P_F)A_k(\bar{y}) + d(\bar{y}), (P_F)A_k(\bar{y}) + d^*(\bar{y})\} = (P_F)A_k(\bar{y}) + d^*(\bar{y})$. Let $a_1, ..., a_{2^k}$ be given as follows:

$$\sum_{u=1}^{2^k} a_u = \sum_{\bar{y} \in \{0,1\}^k} \min_{i=1,..,k} \{P_F A_i(\bar{y}) + d(\bar{y})\} \qquad (20)$$

where $a_u = \min_{i=1,...,k} \{P_F A_i(y_1, ..., y_k) + d(y_1, ..., y_k)\}$ and $u = dec(y_1...y_k) + 1$. Here $dec(y_1...y_k)$ is the decimal value of the binary number $y_1...y_k$. Equations (17), (18), and (19) imply that:

$$\sum_{\bar{y} \in \{0,1\}^k} \min\{P_F A_1(\bar{y}) + d(\bar{y}), ..., P_F A_k(\bar{y}) + d(\bar{y}), P_F A_k(\bar{y}) + d^*(\bar{y})\} = (a_1' - a_1) + \sum_{u=1}^{2^k} a_u$$

where $a_1' = (k-1)(1 - P_D)(1 - P_F)^{k-1} + P_F(1 - P_D)(1 - P_F)^{k-1}, a_1 = (k-1)(1 - P_D)(1 - P_F)^{k-1} + P_D(1 - P_F)^k$. Using the same notation as in (20), $b_1, ..., b_{2^k}$ can be given as follows:

$$\sum_{u=1}^{2^k} b_u = \sum_{\bar{y} \in \{0,1\}^k} \min_{i=1,..,k} \{P_F B_i(\bar{y}) + d(\bar{y})\}$$

Using equation (12) for $\bar{y} = (0,...,0)$ we get:

$$B_k(\bar{y}) \leq B_i(\bar{y}) \leq B_1(\bar{y}), i = 2, .., k-1$$

$$\Rightarrow b_1 = P_F B_k(0,...,0) + d(0,...,0)$$

Let $b_1' = P_F B_{k-1}(0,...,0) + d^*(0,...,0)$. Using the expressions for a_1, a_1', b_1, b_1' we get:

$$(a_1' - a_1) - (b_1' - b_1) = -(P_F)(P_D - P_F)(1 - P_F)^{k-1} \qquad (21)$$

We have two cases here $P_D \geq P_F, P_D(1 - P_D) \geq P_F(1 - P_F)$ and $P_D \geq P_F, P_D(1 - P_D) < P_F(1 - P_F)$.

Case A: $\mathbf{P_D \geq P_F, P_D(1 - P_D) < P_F(1 - P_F)}$

Using equation (12) and $P_D(1 - P_D) < P_F(1 - P_F)$ we get:

$$B_k(\bar{y}) < B_1(\bar{y}) \leq B_i(\bar{y}), \bar{y} = (1, 0, ..., 0)$$

$$\Rightarrow b_{2^{k-1}+1} = P_F B_k(1, 0, ..., 0) + d(1, 0, ..., 0)$$

Let $b'_{2^{k-1}+1} = P_F B_{k-1}(1, 0, ..., 0) + d^*(1, 0, ..., 0)$. Then using $b_{2^{k-1}+1}, b'_{2^{k-1}+1}$ we get:

$$b_{2^{k-1}+1} - b'_{2^{k-1}+1} = P_F(P_D - P_F)(1 - P_F)^{k-1} \qquad (22)$$

Using (21) and (22) we get:

$$(a'_1 - a_1) - (b'_1 - b_1) - (b'_{2^{k-1}+1} - b_{2^{k-1}+1}) = 0$$

These equations imply that:

$$\sum_{\bar{y} \in \{0,1\}^k} \min \left\{ P_F B_1(\bar{y}) + d(\bar{y}), ..., P_F B_{k-1}(\bar{y}) + d(\bar{y}), P_F B_{k-1}(\bar{y}) + d^*(\bar{y}), P_F B_k(\bar{y}) + d(\bar{y}) \right\}$$

$$\leq b'_1 + b'_{2^{k-1}+1} + \sum_{i=2, i \neq 2^{k-1}+1}^{2^k} b_i$$

Therefore we can conclude that:

$$\sum_{\bar{y} \in \{0,1\}^k} \min \left\{ P_F A_1(\bar{y}) + d(\bar{y}), ..., P_F A_k(\bar{y}) + d(\bar{y}), P_F A_k(\bar{y}) + d^*(\bar{y}) \right\} - \sum_{\bar{y} \in \{0,1\}^k} \min \left\{ P_F B_1(\bar{y}) + d(\bar{y}), ..., \right.$$

$$\left. P_F B_{k-1}(\bar{y}) + d(\bar{y}), P_F B_{k-1}(\bar{y}) + d^*(\bar{y}), P_F B_k(\bar{y}) + d(\bar{y}) \right\}$$

$$\geq (a'_1 - a_1) - (b'_1 - b_1) - (b'_{2^{k-1}+1} - b_{2^{k-1}+1}) + \sum_{i=1}^{2^k} a_i - \sum_{i=1}^{2^k} b_i \geq 0$$

$$\Rightarrow (\mathbf{k+1}) \left[\mathbf{P_e}(\underbrace{1, ..., 1}_{\mathbf{k+1}}) - \mathbf{P_e}(\underbrace{2, 1, .., 1, 0}_{\mathbf{k+1}}) \right] \geq \left[(\mathbf{1 - P_F}) \left\{ \sum_{\bar{\mathbf{y}} \in \{0,1\}^k} \min_{\mathbf{i=1,..,k}} \mathbf{A_i}(\bar{\mathbf{y}}) - \sum_{\bar{\mathbf{y}} \in \{0,1\}^k} \min_{\mathbf{i=1,..,k}} \mathbf{B_i}(\bar{\mathbf{y}}) \right\} \right.$$

$$\left. + P_F \left\{ \sum_{\bar{y} \in \{0,1\}^k} \min_{i=1,..,k} A_i(\bar{y}) - \sum_{\bar{y} \in \{0,1\}^k} \min_{i=1,..,k} B_i(\bar{y}) \right\} \right] \geq 0$$

which proves the result for Case A.

Case B: $\mathbf{P_D \geq P_F, P_D(1 - P_D) \geq P_F(1 - P_F)}$

For this case we will modify the induction steps slightly and use the above results and discussions to complete the proof. We will show that:

$$(k+1)\left[P_e(\underbrace{1,...,1}_{k+1}) - P_e(\underbrace{2,1,...,1,0}_{k+1})\right] \geq (P_D - P_F)(1 - P_D - P_F)(1 - P_F)^{k-1}$$

From (8) it is clear that this results holds for M=N=2:

$$2\left(P_e(1,1) - P_e(2,0)\right) = (P_D - P_F)(1 - P_D - P_F)$$

Assume that it holds for M=N=k:

$$(k)\left[P_e(\underbrace{1,...,1}_{k}) - P_e(\underbrace{2,1,...,1,0}_{k})\right] \geq (P_D - P_F)(1 - P_D - P_F)(1 - P_F)^{k-2} \qquad (23)$$

Using equation (12) and $P_D(1 - P_D) \geq P_F(1 - P_F)$:

$$B_1(\bar{y}) \leq B_k(\bar{y}) \leq B_i(\bar{y}), \bar{y} = (1, 0, ..., 0)$$

$$\Rightarrow b_{2^{k-1}+1} = P_F B_1(1, 0, ..., 0) + d(1, 0, ..., 0)$$

Let $b'_{2^{k-1}+1} = P_F B_{k-1}(1, 0, ..., 0) + d^*(1, 0, ..., 0)$. Using the values of $b_{2^{k-1}+1}, b'_{2^{k-1}+1}$ we get:

$$b_{2^{k-1}+1} - b'_{2^{k-1}+1} = P_D P_F (P_D - P_F)(1 - P_F)^{k-2}$$

$$\Rightarrow (a'_1 - a_1) - (b'_1 - b_1) - (b'_{2^{k-1}+1} - b_{2^{k-1}+1}) = P_F(P_D - P_F)(P_D + P_F - 1)(1 - P_F)^{k-2} \quad (24)$$

Consider the observation $(y_1, ..., y_k) = (0, 1, 0, .., 0)$:

$$B_1(\bar{y}) \leq B_k(\bar{y}) \leq B_i(\bar{y}), \bar{y} = (0, 1, 0, ..., 0)$$

$$\Rightarrow b_{2^{k-2}+1} = P_F B_1(0, 1, 0, ..., 0) + d(0, 1, 0, ..., 0) \qquad (25)$$

Let $b'_{2^{k-2}+1} = P_F B_{k-1}(0, 1, 0, ..., 0) + d^*(0, 1, 0, ..., 0)$.

Using (24 and (25) we get:

$$(a_1^{'} - a_1) - (b_1^{'} - b_1) - (b_{2^{k-1}+1}^{'} - b_{2^{k-1}+1}) \; - (b_{2^{k-2}+1}^{'} - b_{2^{k-2}+1}) = P_F(P_D - P_F)(P_D + P_F - 1)(1 - P_F)^{k-2}$$

$$+ P_D P_F(P_D - P_F)(1 - P_F)^{k-2}$$

This implies that:

$$(k+1)\big[P_e(\underbrace{1,...,1}_{k+1}) - P_e(\underbrace{2,1,...,1,0}_{k+1})\big] \geq (1-P_F)\Big\{ \sum_{\bar{y}\in\{0,1\}^k} \big(\min_{i=1,..,k}(A_i(\bar{y})+c(\bar{y})) - \min_{i=1,..,k}(B_i(\bar{y})+c(\bar{y})) \big) \Big\}$$

$$+ (P_F)\Big\{ \sum_{\bar{y}\in\{0,1\}^k} \big(\min_{i=1,..,k}(A_i(\bar{y})+d(\bar{y})) - \min_{i=1,..,k}(B_i(\bar{y})+d(\bar{y})) \big) \Big\} - P_F(P_D-P_F)(1-P_D-P_F)(1-P_F)^{k-2}$$

$$+ P_D P_F(P_D - P_F)(1 - P_F)^{k-2} \tag{26}$$

Using (23), (26), and the fact that $P_D P_F(P_D - P_F)(1 - P_F)^{k-2} \geq 0$ we conclude that:

$$(k+1)\big[P_e(\underbrace{1,...,1}_{k+1}) - P_e(\underbrace{2,1,...,1,0}_{k+1})\big] \geq (P_D - P_F)(1 - P_D - P_F)(1 - P_F)^{k-1} \geq 0$$

This proves the claim for the case $\mathbf{P_D} \geq \mathbf{P_F}$.

A proof for the aforementioned case is sufficient to conclude that the result holds for all values of P_D, P_F. This is due to the fact that for the case $P_D < P_F$ the detector will simply flip the observation bits and the same optimal strategy, which was employed for the case $P_D > P_F$, will be employed by the fusion center. ■

Theorem 4.1 is an extremely useful result since it holds for any values of P_D and P_F. It is quite natural to assume that uniform placement of the sensors should be strictly optimal for sensors which possess a high detection probability and a low probability of false alarm. This results proves that this natural assumption fails for the case when the number of sensors equal the number of placement points.

Next we will prove some properties of the optimal placement on the (P_F, P_D) plane by varying the number of placement points N. We will prove that the optimal solution structure on the (P_F, P_D) plane for $M < N$ remains the same as we increase N.

Theorem 4.2: The sensor-placement point pairs (M, N_1) and (M, N_2), where $M < N_1 < N_2$,

have the same optimal placement structure on the (P_F, P_D) plane.

Proof: We will use the notation $P_e(v_1, ..., v_M)\big|_{N_i}, i = 1, 2$ to distinguish between the two situations.

First consider the case (M, N_1) where we have M agents and N_1 placement points. For a specific placement $\bar{v} = (v_1, ..., v_M) \in \Lambda^M$,

$$P_e(v_1, ..., v_M)\big|_{N_1} = \frac{1}{N_1} \sum_{\bar{y} \in \{0,1\}^M} \min_{i=1,..,N_1} \{ \sum_{j=1, j \neq i}^{N_1} p_j(\bar{y}, \bar{v}) \} \qquad (27)$$

where $p_j(\bar{y}, \bar{v})$ is given by equation (1). Now $v_j = 0, j = (M+1), ..., N_1$, therefore we can take $p_j(\bar{y}, \bar{v}) = p_{M+1}(\bar{y}), j = (M+1), ..., N_1$. Note that $p_{M+1}(\bar{y})$ does not depend on the placement vector $\bar{v}$. Using this fact (27) can be updated as follows:

$$P_e(v_1, ..., v_M)\big|_{N_1} = \frac{1}{N_1} \sum_{\bar{y} \in \{0,1\}^M} \min_{i=1,..,M+1} \left[\sum_{j=1, j \neq i}^{M} p_j(\bar{y}, \bar{v}) + (N_1 - M) p_{M+1}(\bar{y}) \chi_{(i \neq M+1)} \right.$$

$$\left. + (N_1 - M - 1) p_{M+1}(\bar{y}) \chi_{(i=M+1)} \right] \qquad (28)$$

Consider another placement vector $\hat{\bar{v}} = (\hat{v}_1, ..., \hat{v}_M) \in \Lambda^M$.

$$P_e(\hat{v}_1, ..., \hat{v}_M)\big|_{N_1} = \frac{1}{N_1} \sum_{\bar{y} \in \{0,1\}^M} \min_{i=1,..,M+1} \left[\sum_{j=1, j \neq i}^{M} p_j(\bar{y}, \hat{\bar{v}}) + (N_1 - M) p_{M+1}(\bar{y}) \chi_{(i \neq M+1)} \right.$$

$$\left. + (N_1 - M - 1) p_{M+1}(\bar{y}) \chi_{(i=M+1)} \right] \qquad (29)$$

Both (N_1, M) and (N_2, M) have the same set of admissible placements, Λ^M. $P_e(v_1, ..., v_M)\big|_{N_2}, P_e(\hat{v}_1, ..., \hat{v}_M)\big|_{N_2}$ can be written as:

$$P_e(v_1, ..., v_M)\big|_{N_2} = \frac{1}{N_2} \sum_{\bar{y} \in \{0,1\}^M} \min_{i=1,..,M+1} \left[\sum_{j=1, j \neq i}^{M} p_j(\bar{y}, \bar{v}) + (N_1 - M) p_{M+1}(\bar{y}) \chi_{(i \neq M+1)} + \right.$$

$$\left. (N_1 - M - 1) p_{M+1}(\bar{y}) \chi_{(i=M+1)} + (N_2 - N_1) p_{M+1}(\bar{y}) \chi_{(i \neq M+1)} + (N_2 - N_1) p_{M+1}(\bar{y}) \chi_{(i=M+1)} \right] \quad (30)$$

$$P_e(\hat{v}_1, ..., \hat{v}_M)\big|_{N_2} = \frac{1}{N_2} \sum_{\bar{y} \in \{0,1\}^M} \min_{i=1,..,M+1} \left[\sum_{j=1, j \neq i}^{M} p_j(\bar{y}, \hat{\bar{v}}) + (N_1 - M) p_{M+1}(\bar{y}) \chi_{(i \neq M+1)} + \right.$$

$$(N_1 - M - 1)p_{M+1}(\bar{y})\chi_{(i=M+1)} + (N_2 - N_1)p_{M+1}(\bar{y})\chi_{(i \neq M+1)} + (N_2 - N_1)p_{M+1}(\bar{y})\chi_{(i=M+1)} \Big] \quad (31)$$

Equations (28), (29), (30), and (31) imply that:

$$N_2 \times P_e(v_1, ..., v_M)\big|_{N_2} - N_2 \times P_e(\hat{v}_1, ..., \hat{v}_M)\big|_{N_2} = N_1 \times P_e(v_1, ..., v_M)\big|_{N_1} - N_1 \times P_e(\hat{v}_1, ..., \hat{v}_M)\big|_{N_1}$$

$$(32)$$

Since $\bar{v}$ and $\hat{\bar{v}}$ are two arbitrarily chosen placements from the set Λ^M, (32) implies that as we transition from (M, N_1) to (M, N_2) the comparison equations which determine the optimal placements do not change. Therefore, we conclude that both (M, N_1) and (M, N_2) have the same optimal placement structure on the (P_F, P_D) plane. ∎

Theorem 4.2 has important practical implications. It implies that if we are given a sensor-placement pair (M, N) where $M < N$ we only need to analyze the sensor-placement pair $(M, M+1)$ and its optimal placements will be valid for all $N > M$. This results in a significant reduction in computational complexity.

Corollary 4.1: Let $\boldsymbol{\Delta}^{M,N_1}$ be the set of strictly optimal placements for the sensor-placement pair $(M, N_1), N_1 = M$. Then $\boldsymbol{\Delta}^{M,N_2} = \boldsymbol{\Delta}^{M,N_1} \cup \big\{(v_1, ..., v_M) = (1, ..., 1)\big\}$ is the set of strictly optimal placements for the sensor-placement pair $(M, N_2), N_2 > M$.

Corollary 4.1 follows directly from Theorem 4.1, Theorem 4.2, and some basic facts regarding the uniform placement of sensors. One can easily recognize the fact that uniform placement of sensors will always belong to the set of strictly optimal placements $\boldsymbol{\Delta}^{M,N}$, for a sensor-placement pair $(M, N), M < N$. This can be established by considering extremely high values of P_D and extremely low values of P_F.

V. A MAJORIZATION APPROACH

In this section, we will utilize concepts from Majorization Theory to characterize some properties of the optimal solution. We will utilize the notion of a majorization based placement scale defined in Section II.

It should be noted that majorization is a partial ordering and therefore not all elements of Λ^M can be placed on a majorization based scale. There can be many sets of optimal placements dependent on what we choose as the optimal placement at points where no placement is strictly optimal. For a given (M, N), an important question is the existence of a set of optimal

placements which can be placed on a majorization based scale.

Conjecture 5.1: For any (M, N), there exists a set of optimal placements $O^M \subseteq \Lambda^M$ which can be placed on a majorization based placement scale.

We explain the intution behind this conjecture by considering the sensor-placement pair, $(M, N) = (6, 6)$, which has Λ^M of size $|\Lambda^M| = 11$. Note that the placements $(4, 1, 1)$ and $(3, 3)$ are not comparable with respect to majorization. Similarly $(3, 1, 1, 1)$ and $(2, 2, 2)$ are not comparable. The conjecture would be false if every set of optimal placements contained both $(4, 1, 1)$ and $(3, 3)$ or contained both $(3, 1, 1, 1)$ and $(2, 2, 2)$. $\{(6, 0), (5, 1), (4, 2), (3, 2, 1), (2, 2, 1, 1), (2, 1, 1, 1, 1)\}$ is a set of optimal placements for this case. It should be noted that for high values of P_F, $P_D \geq P_F$, the optimal placements tend to have large number of sensors at a small number of points (concentrated placement) and for low values of P_F the optimal placements tend to have small number of sensors placed at a higher number of points (spread out placement). In this case the placements $(4, 2), (3, 2, 1)$, and $(2, 2, 1, 1)$ are optimal in the regions where one would have naturally expected the placements $(3, 1, 1, 1), (2, 2, 2), (4, 1, 1)$, and $(3, 3)$ to be optimal. The aforementioned example indicates existence of the following properties of non-comparable placements which form the basis of this conjecture:

- Placements containing sensors concentrated at few points but not enough to outperform other placements, for higher values of P_F, which are highly concentrated and can be placed on a majorization scale.

- Placements containing sensors that are relatively spread out but not enough to outperform other placements, for lower values of P_F, which have a higher spread of and can be placed on a majorization based scale.

- Placements not possesing a fine balance between concentration and spread of sensors to be optimal for values of P_F that are neither high nor low.

The main result of this section can be stated as follows:

Proposition 5.1: By fixing P_D and increasing P_F, for $M \leq 5$ and $P_D \geq P_F$, the optimal placement occurs non decreasingly on a majorization based placement scale.

Proof: We will only consider the sensor-placement pair $(M, N) = (4, 4)$. Other cases where $M \leq 5$ can be proved similarly.

The admissible placement set is given as $\Lambda^M = \{(1,1,1,1), (2,1,1,0), (2,2,0,0), (3,1,0,0), (4,0,0,0)\}$. Since $M = N$ we do not need to consider the placement $(1,1,1,1)$. Using majorization theory the following majorization based placement scale can be defined:

$$(4,0,0,0) \succ (3,1,0,0) \succ (2,2,0,0) \succ (2,1,1,0) \tag{33}$$

Using (5) we will derive the regions on the (P_b, P_a) plane on which these placements are optimal, respectively. First we provide the conditional probabilties $p_1(y_1, y_2, y_3, y_4)\big|_{(v_1, v_2, v_3, v_4)}$ for the sensor placement $(v_1, v_2, v_3, v_4) = (2, 1, 1, 0)$ using equation (5) and (6).

$$p_1(1,1,1,1)\big|_{(2,1,1,0)} = P_D^2 P_F^2 \ , \ p_1(1,1,1,0)\big|_{(2,1,1,0)} = P_D^2 P_F (1 - P_F)$$

$$p_1(1,1,0,1)\big|_{(2,1,1,0)} = P_D^2 P_F (1 - P_F) \ , \ p_1(1,1,0,0)\big|_{(2,1,1,0)} = P_D^2 (1 - P_F)^2$$

$$p_1(1,0,1,1)\big|_{(2,1,1,0)} = P_D P_F^2 (1 - P_D) \ , \ p_1(1,0,1,0)\big|_{(2,1,1,0)} = P_D P_F (1 - P_D)(1 - P_F)$$

$$p_1(1,0,0,1)\big|_{(2,1,1,0)} = P_D P_F (1 - P_D)(1 - P_F) \ , \ p_1(1,0,0,0)\big|_{(2,1,1,0)} = P_D (1 - P_D)(1 - P_F)^2$$

$$p_1(0,1,1,1)\big|_{(2,1,1,0)} = P_D P_F^2 (1 - P_D) \ , \ p_1(0,1,1,0)\big|_{(2,1,1,0)} = P_D P_F (1 - P_D)(1 - P_F)$$

$$p_1(0,1,0,1)\big|_{(2,1,1,0)} = P_D P_F (1 - P_D)(1 - P_F) \ , \ p_1(0,1,0,0)\big|_{(2,1,1,0)} = P_D (1 - P_D)(1 - P_F)^2$$

$$p_1(0,0,1,1)\big|_{(2,1,1,0)} = P_F^2 (1 - P_D)^2 \ , \ p_1(0,0,1,0)\big|_{(2,1,1,0)} = P_F (1 - P_F)(1 - P_D)^2$$

$$p_1(0,0,0,1)\big|_{(2,1,1,0)} = P_F (1 - P_F)(1 - P_D)^2 \ , \ p_1(0,0,0,0)\big|_{(2,1,1,0)} = (1 - P_D)^2 (1 - P_F)^2$$

The probabilities $p_2(y_1, y_2, y_3, y_4)\big|_{(2,1,1,0)}$ are given as follows:

$$p_2(1,1,1,1)\big|_{(2,1,1,0)} = P_D P_F^3 \ , \ p_2(1,1,1,0)\big|_{(2,1,1,0)} = P_D P_F^2 (1 - P_F)$$

$$p_2(1,1,0,1)\big|_{(2,1,1,0)} = P_F^3 (1 - P_D) \ , \ p_2(1,1,0,0)\big|_{(2,1,1,0)} = P_F^2 (1 - P_D)(1 - P_F)$$

$$p_2(1,0,1,1)\big|_{(2,1,1,0)} = P_D P_F^2 (1 - P_F) \ , \ p_2(1,0,1,0)\big|_{(2,1,1,0)} = P_D P_F (1 - P_F)^2$$

$$p_2(1,0,0,1)\big|_{(2,1,1,0)} = P_F^2(1-P_D)(1-P_F) \ , \ p_2(1,0,0,0)\big|_{(2,1,1,0)} = P_F(1-P_D)(1-P_F)^2$$

$$p_2(0,1,1,1)\big|_{(2,1,1,0)} = P_D P_F^2(1-P_F) \ , \ p_2(0,1,1,0)\big|_{(2,1,1,0)} = P_D P_F(1-P_F)^2$$

$$p_2(0,1,0,1)\big|_{(2,1,1,0)} = P_F^2(1-P_D)(1-P_F) \ , \ p_2(0,1,0,0)\big|_{(2,1,1,0)} = P_F(1-P_D)(1-P_F)^2$$

$$p_2(0,0,1,1)\big|_{(2,1,1,0)} = P_D P_F(1-P_F)^2 \ , \ p_2(0,0,1,0)\big|_{(2,1,1,0)} = P_D(1-P_F)^3$$

$$p_2(0,0,0,1)\big|_{(2,1,1,0)} = P_F(1-P_D)(1-P_F)^2 \ , \ p_2(0,0,0,0)\big|_{(2,1,1,0)} = (1-P_D)(1-P_F)^3$$

The probabilities $p_3(y_1, y_2, y_3, y_4)\big|_{(2,1,1,0)}$ are given as follows:

$$p_3(1,1,1,1)\big|_{(2,1,1,0)} = P_D P_F^3 \ , \ p_3(1,1,1,0)\big|_{(2,1,1,0)} = P_F^3(1-P_D)$$

$$p_3(1,1,0,1)\big|_{(2,1,1,0)} = P_D P_F^2(1-P_F) \ , \ p_3(1,1,0,0)\big|_{(2,1,1,0)} = P_F^2(1-P_D)(1-P_F)$$

$$p_3(1,0,1,1)\big|_{(2,1,1,0)} = P_D P_F^2(1-P_F) \ , \ p_3(1,0,1,0)\big|_{(2,1,1,0)} = P_F^2(1-P_D)(1-P_F)$$

$$p_3(1,0,0,1)\big|_{(2,1,1,0)} = P_D P_F(1-P_F)^2 \ , \ p_3(1,0,0,0)\big|_{(2,1,1,0)} = P_F(1-P_D)(1-P_F)^2$$

$$p_3(0,1,1,1)\big|_{(2,1,1,0)} = P_D P_F^2(1-P_F) \ , \ p_3(0,1,1,0)\big|_{(2,1,1,0)} = P_F^2(1-P_D)(1-P_F)$$

$$p_3(0,1,0,1)\big|_{(2,1,1,0)} = P_D P_F(1-P_F)^2 \ , \ p_3(0,1,0,0)\big|_{(2,1,1,0)} = P_F(1-P_D)(1-P_F)^2$$

$$p_3(0,0,1,1)\big|_{(2,1,1,0)} = P_D P_F(1-P_F)^2 \ , \ p_3(0,0,1,0)\big|_{(2,1,1,0)} = P_F(1-P_D)(1-P_F)^2$$

$$p_3(0,0,0,1)\big|_{(2,1,1,0)} = P_D(1-P_F)^3 \ , \ p_3(0,0,0,0)\big|_{(2,1,1,0)} = (1-P_D)(1-P_F)^3$$

The probabilities $p_4(y_1, y_2, y_3, y_4)\big|_{(2,1,1,0)}$ are given as follows:

$$p_4(1,1,1,1)\big|_{(2,1,1,0)} = P_F^4 \ , \ p_4(1,1,1,0)\big|_{(2,1,1,0)} = P_F^3(1-P_F)$$

$$p_4(1,1,0,1)\big|_{(2,1,1,0)} = P_F^3(1-P_F) \ , \ p_4(1,1,0,0)\big|_{(2,1,1,0)} = P_F^2(1-P_F)^2$$

$$p_4(1,0,1,1)\big|_{(2,1,1,0)} = P_F^3(1-P_F) \ , \ p_4(1,0,1,0)\big|_{(2,1,1,0)} = P_F^2(1-P_F)^2$$

$$p_4(1,0,0,1)\big|_{(2,1,1,0)} = P_F^2(1-P_F)^2 \ , \ p_4(1,0,0,0)\big|_{(2,1,1,0)} = P_F(1-P_F)^3$$

$$p_4(0,1,1,1)\big|_{(2,1,1,0)} = P_F^3(1 - P_F) \ , \ \ p_4(0,1,1,0)\big|_{(2,1,1,0)} = P_F^2(1 - P_F)^2$$

$$p_4(0,1,0,1)\big|_{(2,1,1,0)} = P_F^2(1 - P_F)^2 \ , \ \ p_4(0,1,0,0)\big|_{(2,1,1,0)} = P_F(1 - P_F)^3$$

$$p_4(0,0,1,1)\big|_{(2,1,1,0)} = P_F^2(1 - P_F)^2 \ , \ \ p_4(0,0,1,0)\big|_{(2,1,1,0)} = P_F(1 - P_F)^3$$

$$p_4(0,0,0,1)\big|_{(2,1,1,0)} = P_F(1 - P_F)^3 \ , \ \ p_4(0,0,0,0)\big|_{(2,1,1,0)} = (1 - P_F)^4$$

Let us denote $I_i(\bar{y})$ by:

$$I_i(y_1, y_2, y_3, y_4)\big|_{(v_1,v_2,v_3,v_4)} = \sum_{j=1, j \neq i} p_j(y_1, y_2, y_3, y_4)\big|_{(v_1,v_2,v_3,v_4)} \tag{34}$$

Using equation (34) and the conditional probabilities we get the following values for $I_i(y_1, y_2, y_3, y_4)\big|_{(2,1,1,0)}$:

$$I_1(1,1,1,1)\big|_{(2,1,1,0)} = 2P_D P_F^3 + P_F^4$$

$$I_2(1,1,1,1)\big|_{(2,1,1,0)} = P_D^2 P_F^2 + P_D P_F^3 + P_F^4$$

$$I_3(1,1,1,1)\big|_{(2,1,1,0)} = P_D^2 P_F^2 + P_D P_F^3 + P_F^4$$

$$I_4(1,1,1,1)\big|_{(2,1,1,0)} = P_D^2 P_F^2 + 2P_D P_F^3$$

$$I_1(1,1,1,0)\big|_{(2,1,1,0)} = P_D P_F^2(1 - P_F) + P_F^3(1 - P_D) + P_F^3(1 - P_F)$$

$$I_2(1,1,1,0)\big|_{(2,1,1,0)} = P_D^2 P_F(1 - P_F) + P_F^3(1 - P_D) + P_F^3(1 - P_F)$$

$$I_3(1,1,1,0)\big|_{(2,1,1,0)} = P_D^2 P_F(1 - P_F) + P_D P_F^2(1 - P_F) + P_F^3(1 - P_F)$$

$$I_4(1,1,1,0)\big|_{(2,1,1,0)} = P_D^2(1-P_F) + P_D P_F^2(1-P_F) + P_F^3(1-P_D)$$

$$I_1(1,1,0,1)\big|_{(2,1,1,0)} = P_D^3(1-P_D) + P_D P_F^2(1-P_F) + P_F^3(1-P_F)$$

$$I_2(1,1,0,1)\big|_{(2,1,1,0)} = P_D^2 P_F(1-P_F) + P_D P_F^2(1-P_F) + P_F^3(1-P_F)$$

$$I_3(1,1,0,1)\big|_{(2,1,1,0)} = P_D^2 P_F(1-P_F) + P_F^3(1-P_D) + P_F^3(1-P_F)$$

$$I_4(1,1,0,1)\big|_{(2,1,1,0)} = P_D^2 P_F(1-P_F) + P_F^3(1-P_D) + P_D P_F^2(1-P_F)$$

$$I_1(1,1,0,0)\big|_{(2,1,1,0)} = 2P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_2(1,1,0,0)\big|_{(2,1,1,0)} = P_D^2(1-P_F)^2 + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_3(1,1,0,0)\big|_{(2,1,1,0)} = P_D^2(1-P_F)^2 + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_4(1,1,0,0)\big|_{(2,1,1,0)} = P_D^2(1-P_F)^2 + 2P_F^2(1-P_D)(1-P_F)$$

$$I_1(1,0,1,1)\big|_{(2,1,1,0)} = 2P_D P_F^2(1-P_F) + P_F^3(1-P_F)$$

$$I_2(1,0,1,1)\big|_{(2,1,1,0)} = P_D P_F^2(1-P_D) + P_D P_F^2(1-P_F) + P_F^3(1-P_F)$$

$$I_3(1,0,1,1)\big|_{(2,1,1,0)} = P_D P_F^2(1-P_D) + P_D P_F^2(1-P_F) + P_F^3(1-P_F)$$

$$I_4(1,0,1,1)\big|_{(2,1,1,0)} = P_D P_F^2(1-P_D) + 2P_D P_F^2(1-P_F)$$

$$I_1(1,0,1,0)\big|_{(2,1,1,0)} = P_D P_F(1-P_F)^2 + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_2(1,0,1,0)\big|_{(2,1,1,0)} = P_D P_F(1-P_D)(1-P_F) + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_3(1,0,1,0)\big|_{(2,1,1,0)} = P_D P_F(1-P_D)(1-P_F) + P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2$$

$$I_4(1,0,1,0)\big|_{(2,1,1,0)} = P_D P_F(1-P_D)(1-P_F) + P_D P_F(1-P_F)^2 + P_F^2(1-P_D)(1-P_F)$$

$$I_1(1,0,0,1)\big|_{(2,1,1,0)} = P_F^2(1-P_D)(1-P_F) + P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2$$

$$I_2(1,0,0,1)\big|_{(2,1,1,0)} = P_D P_F(1-P_D)(1-P_F) + P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2$$

$$I_3(1,0,0,1)\big|_{(2,1,1,0)} = P_D P_F(1-P_D)(1-P_F) + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_4(1,0,0,1)\big|_{(2,1,1,0)} = P_D P_F(1-P_D)(1-P_F) + P_F^2(1-P_D)(1-P_F) + P_D P_F(1-P_F)^2$$

$$I_1(1,0,0,0)\big|_{(2,1,1,0)} = 2P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3$$

$$I_2(1,0,0,0)\big|_{(2,1,1,0)} = P_D(1-P_D)(1-P_F)^2 + P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3$$

$$I_3(1,0,0,0)\big|_{(2,1,1,0)} = P_D(1-P_D)(1-P_F)^2 + P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3$$

$$I_4(1,0,0,0)\big|_{(2,1,1,0)} = P_D(1-P_D)(1-P_F)^2 + 2P_F(1-P_D)(1-P_F)^2$$

$$I_1(0,1,1,1)\big|_{(2,1,1,0)} = 2P_D P_F^2(1-P_F) + P_F^3(1-P_F)$$

$$I_2(0,1,1,1)\big|_{(2,1,1,0)} = P_D P_F^2(1-P_D) + P_D P_F^2(1-P_F) + P_F^3(1-P_F)$$

$$I_3(0,1,1,1)\big|_{(2,1,1,0)} = P_D P_F^2(1-P_D) + P_D P_F^2(1-P_F) + P_F^3(1-P_F)$$

$$I_4(0,1,1,1)\big|_{(2,1,1,0)} = P_D P_F^2(1-P_D) + 2P_D P_F^2(1-P_F)$$

$$I_1(0,1,1,0)\big|_{(2,1,1,0)} = P_D P_F(1-P_F)^2 + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_2(0,1,1,0)\big|_{(2,1,1,0)} = P_D P_F(1-P_D)(1-P_F) + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_3(0,1,1,0)\big|_{(2,1,1,0)} = P_D P_F(1-P_D)(1-P_F) + P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2$$

$$I_4(0,1,1,0)\big|_{(2,1,1,0)} = P_D P_F(1-P_D)(1-P_F) + P_D P_F(1-P_F)^2 + P_F^2(1-P_D)(1-P_F)$$

$$I_1(0,1,0,1)\big|_{(2,1,1,0)} = P_F^2(1-P_D)(1-P_F) + P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2$$

$$I_2(0,1,0,1)\big|_{(2,1,1,0)} = P_D P_F(1-P_D)(1-P_F) + P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2$$

$$I_3(0,1,0,1)\big|_{(2,1,1,0)} = P_D P_F(1-P_D)(1-P_F) + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_4(0,1,0,1)\big|_{(2,1,1,0)} = P_D P_F(1-P_D)(1-P_F) + P_F^2(1-P_D)(1-P_F) + P_D P_F(1-P_F)^2$$

$$I_1(0,1,0,0)\big|_{(2,1,1,0)} = 2P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3$$

$$I_2(0,1,0,0)\big|_{(2,1,1,0)} = P_D(1-P_D)(1-P_F)^2 + P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3$$

$$I_3(0,1,0,0)\big|_{(2,1,1,0)} = P_D(1-P_D)(1-P_F)^2 + P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3$$

$$I_4(0,1,0,0)\big|_{(2,1,1,0)} = P_D(1-P_D)(1-P_F)^2 + 2P_F(1-P_D)(1-P_F)^2$$

$$I_1(0,0,1,1)\big|_{(2,1,1,0)} = 2P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2$$

$$I_2(0,0,1,1)\big|_{(2,1,1,0)} = P_F^2(1-P_D)^2 + P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2$$

$$I_3(0,0,1,1)\big|_{(2,1,1,0)} = P_F^2(1-P_D)^2 + P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2$$

$$I_4(0,0,1,1)\big|_{(2,1,1,0)} = P_F^2(1-P_D)^2 + 2P_D P_F(1-P_F)^2$$

$$I_1(0,0,1,0)\big|_{(2,1,1,0)} = P_D(1-P_F)^3 + P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3$$

$$I_2(0,0,1,0)\big|_{(2,1,1,0)} = P_F(1-P_F)(1-P_D)^2 + P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3$$

$$I_3(0,0,1,0)\big|_{(2,1,1,0)} = P_F(1-P_F)(1-P_D)^2 + P_D(1-P_F)^3 + P_F(1-P_F)^3$$

$$I_4(0,0,1,0)\big|_{(2,1,1,0)} = P_F(1-P_F)(1-P_D)^2 + P_D(1-P_F)^3 + P_F(1-P_D)(1-P_F)^2$$

$$I_1(0,0,0,1)\big|_{(2,1,1,0)} = P_F(1-P_D)(1-P_F)^2 + P_D(1-P_F)^3 + P_F(1-P_F)^3$$

$$I_2(0,0,0,1)\big|_{(2,1,1,0)} = P_F(1-P_F)(1-P_D)^2 + P_D(1-P_F)^3 + P_F(1-P_F)^3$$

$$I_3(0,0,0,1)\big|_{(2,1,1,0)} = P_F(1-P_F)(1-P_D)^2 + P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3$$

$$I_4(0,0,0,1)\big|_{(2,1,1,0)} = P_F(1-P_F)(1-P_D)^2 + P_F(1-P_D)(1-P_F)^2 + P_D(1-P_F)^3$$

$$I_1(0,0,0,0)\big|_{(2,1,1,0)} = 2(1-P_D)(1-P_F)^3 + (1-P_F)^4$$

$$I_2(0,0,0,0)\big|_{(2,1,1,0)} = (1-P_D)^2(1-P_F)^2 + (1-P_D)(1-P_F)^3 + (1-P_F)^4$$

$$I_3(0,0,0,0)\big|_{(2,1,1,0)} = (1-P_D)^2(1-P_F)^2 + (1-P_D)(1-P_F)^3 + (1-P_F)^4$$

$$I_4(0,0,0,0)\big|_{(2,1,1,0)} = (1-P_D)^2(1-P_F)^2 + 2(1-P_D)(1-P_F)^3 \tag{35}$$

Now we compute the conditional probabilties $p_1(y_1,y_2,y_3,y_4)\big|_{(v_1,v_2,v_3,v_4)}$ for the sensor placement $(v_1,v_2,v_3,v_4) = (2,2,0,0)$ using equation (5) and (6).

$$p_1(1,1,1,1)\big|_{(2,2,0,0)} = P_D^2 P_F^2 \ , \ p_1(1,1,1,0)\big|_{(2,2,0,0)} = P_D^2 P_F(1-P_F)$$

$$p_1(1,1,0,1)\big|_{(2,2,0,0)} = P_D^2 P_F(1-P_F) \ , \ p_1(1,1,0,0)\big|_{(2,2,0,0)} = P_D^2(1-P_F)^2$$

$$p_1(1,0,1,1)\big|_{(2,2,0,0)} = P_D(P_F^2)(1-P_F) \ , \ p_1(1,0,1,0)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F)$$

$$p_1(1,0,0,1)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F) \ , \ p_1(1,0,0,0)\big|_{(2,2,0,0)} = P_D(1-P_D)(1-P_F)^2$$

$$p_1(0,1,1,1)\big|_{(2,2,0,0)} = P_D(P_F^2)(1-P_D) \ , \ p_1(0,1,1,0)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F)$$

$$p_1(0,1,0,1)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F) \ , \ p_1(0,1,0,0)\big|_{(2,2,0,0)} = P_D(1-P_D)(1-P_F)^2$$

$$p_1(0,0,1,1)\big|_{(2,2,0,0)} = P_F^2(1-P_D)^2 \ , \ p_1(0,0,1,0)\big|_{(2,2,0,0)} = P_F(1-P_F)(1-P_D)^2$$

$$p_1(0,0,0,1)\big|_{(2,2,0,0)} = P_F(1-P_F)(1-P_D)^2 \ , \ p_1(0,0,0,0)\big|_{(2,2,0,0)} = (1-P_D)^2(1-P_F)^2$$

The conditional probabilities $p_2(y_1,y_2,y_3,y_4)\big|_{(2,2,0,0)}$ are given as follows:

$$p_2(1,1,1,1)\big|_{(2,2,0,0)} = P_D^2 P_F^2 \ , \ p_2(1,1,1,0)\big|_{(2,2,0,0)} = P_D P_F^2(1-P_D)$$

$$p_2(1,1,0,1)\big|_{(2,2,0,0)} = P_D P_F^2(1-P_D) \ , \ \ p_2(1,1,0,0)\big|_{(2,2,0,0)} = P_F^2(1-P_D)^2$$

$$p_2(1,0,1,1)\big|_{(2,2,0,0)} = P_D^2 P_F(1-P_F) \ , \ \ p_2(1,0,1,0)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F)$$

$$p_2(1,0,0,1)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F) \ , \ \ p_2(1,0,0,0)\big|_{(2,2,0,0)} = P_F(1-P_F)(1-P_D)^2$$

$$p_2(0,1,1,1)\big|_{(2,2,0,0)} = P_D^2 P_F(1-P_F) \ , \ \ p_2(0,1,1,0)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F)$$

$$p_2(0,1,0,1)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F) \ , \ \ p_2(0,1,0,0)\big|_{(2,2,0,0)} = P_F(1-P_F)(1-P_D)^2$$

$$p_2(0,0,1,1)\big|_{(2,2,0,0)} = P_D^2(1-P_F)^2 \ , \ \ p_2(0,0,1,0)\big|_{(2,2,0,0)} = P_D(1-P_D)(1-P_F)^2$$

$$p_2(0,0,0,1)\big|_{(2,2,0,0)} = P_D(1-P_D)(1-P_F)^2 \ , \ \ p_2(0,0,0,0)\big|_{(2,2,0,0)} = (1-P_D)^2(1-P_F)^2$$

The conditional probabilities $p_3(y_1,y_2,y_3,y_4)\big|_{(2,2,0,0)}$ are given as follows:

$$p_3(1,1,1,1)\big|_{(2,2,0,0)} = P_F^4 \ , \ \ p_3(1,1,1,0)\big|_{(2,2,0,0)} = P_F^3(1-P_F)$$

$$p_3(1,1,0,1)\big|_{(2,2,0,0)} = P_F^3(1-P_F) \ , \ \ p_3(1,1,0,0)\big|_{(2,2,0,0)} = P_F^2(1-P_F)^2$$

$$p_3(1,0,1,1)\big|_{(2,2,0,0)} = P_F^3(1-P_F) \ , \ \ p_3(1,0,1,0)\big|_{(2,2,0,0)} = P_F^2(1-P_F)^2$$

$$p_3(1,0,0,1)\big|_{(2,2,0,0)} = P_F^2(1-P_F)^2 \ , \ \ p_3(1,0,0,0)\big|_{(2,2,0,0)} = P_F(1-P_F)^3$$

$$p_3(0,1,1,1)\big|_{(2,2,0,0)} = P_F^3(1-P_F) \ , \ \ p_3(0,1,1,0)\big|_{(2,2,0,0)} = P_F^2(1-P_F)^2$$

$$p_3(0,1,0,1)\big|_{(2,2,0,0)} = P_F^2(1-P_F)^2 \ , \ \ p_3(0,1,0,0)\big|_{(2,2,0,0)} = P_F(1-P_F)^3$$

$$p_3(0,0,1,1)\big|_{(2,2,0,0)} = P_F^2(1-P_F)^2 \ , \ \ p_3(0,0,1,0)\big|_{(2,2,0,0)} = P_F(1-P_F)^3$$

$$p_3(0,0,0,1)\big|_{(2,2,0,0)} = P_F(1-P_F)^3 \ , \ \ p_3(0,0,0,0)\big|_{(2,2,0,0)} = (1-P_F)^4$$

It should be noted that in this case $p_3(y_1,y_2,y_3,y_4)\big|_{(2,2,0,0)} = p_4(y_1,y_2,y_3,y_4)\big|_{(2,2,0,0)}$. Using equation (34) and the conditional probabilities we get the following values for $I_i(y_1,y_2,y_3,y_4)\big|_{(2,2,0,0)}$:

$$I_1(1,1,1,1)\big|_{(2,2,0,0)} = P_D^2 P_F^2 + 2P_F^4$$

$$I_2(1,1,1,1)\big|_{(2,2,0,0)} = P_D^2 P_F^2 + 2P_F^4$$

$$I_3(1,1,1,1)\big|_{(2,2,0,0)} = 2P_D^2 P_F^2 + P_F^4$$

$$I_1(1,1,1,0)\big|_{(2,2,0,0)} = P_D P_F^2(1-P_D) + 2P_F^3(1-P_F)$$

$$I_2(1,1,1,0)\big|_{(2,2,0,0)} = P_D^2 P_F(1-P_F) + 2P_F^3(1-P_F)$$

$$I_3(1,1,1,0)\big|_{(2,2,0,0)} = P_D^2 P_F(1-P_F) + P_D P_F^2(1-P_D) + P_F^3(1-P_F)$$

$$I_1(1,1,0,1)\big|_{(2,2,0,0)} = P_D P_F^2(1-P_D) + 2P_F^3(1-P_F)$$

$$I_2(1,1,0,1)\big|_{(2,2,0,0)} = P_D^2 P_F(1-P_F) + 2P_F^3(1-P_F)$$

$$I_3(1,1,0,1)\big|_{(2,2,0,0)} = P_D^2 P_F(1-P_F) + P_D P_F^2(1-P_D) + P_F^3(1-P_F)$$

$$I_1(1,1,0,0)\big|_{(2,2,0,0)} = P_F^2(1-P_D)^2 + 2P_F^2(1-P_F)^2$$

$$I_2(1,1,0,0)\big|_{(2,2,0,0)} = P_D^2(1-P_F)^2 + 2P_F^2(1-P_F)^2$$

$$I_3(1,1,0,0)\big|_{(2,2,0,0)} = P_D^2(1-P_F)^2 + P_F^2(1-P_D)^2 + P_F^2(1-P_F)^2$$

$$I_1(1,0,1,1)\big|_{(2,2,0,0)} = P_D^2 P_F(1-P_F) + 2P_F^3(1-P_F)$$

$$I_2(1,0,1,1)\big|_{(2,2,0,0)} = P_D P_F^2(1-P_D) + 2P_F^3(1-P_F)$$

$$I_3(1,0,1,1)\big|_{(2,2,0,0)} = P_D P_F^2(1-P_F) + P_F^2 P_F(1-P_F) + P_F^3(1-P_F)$$

$$I_1(1,0,1,0)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2$$

$$I_2(1,0,1,0)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2$$

$$I_3(1,0,1,0)\big|_{(2,2,0,0)} = 2P_D P_F(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_1(1,0,0,1)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2$$

$$I_2(1,0,0,1)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2$$

$$I_3(1,0,0,1)\big|_{(2,2,0,0)} = 2P_D P_F(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_1(1,0,0,0)\big|_{(2,2,0,0)} = P_F(1-P_F)(1-P_D)^2 + 2P_F(1-P_F)^3$$

$$I_2(1,0,0,0)\big|_{(2,2,0,0)} = P_D(1-P_D)(1-P_F)^2 + 2P_F(1-P_F)^3$$

$$I_3(1,0,0,0)\big|_{(2,2,0,0)} = P_D(1-P_D)(1-P_F)^2 + P_F(1-P_F)(1-P_D)^2 + P_F(1-P_F)^3$$

$$I_1(0,1,1,1)\big|_{(2,2,0,0)} = P_D^2 P_F(1-P_F) + 2P_F^3(1-P_F)$$

$$I_2(0,1,1,1)\big|_{(2,2,0,0)} = P_D P_F^2(1-P_D) + 2P_F^3(1-P_F)$$

$$I_3(0,1,1,1)\big|_{(2,2,0,0)} = P_D P_F^2(1-P_D) + P_D^2 P_F(1-P_F) + P_F^3(1-P_F)$$

$$I_1(0,1,1,0)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2$$

$$I_2(0,1,1,0)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2$$

$$I_3(0,1,1,0)\big|_{(2,2,0,0)} = 2P_D P_F(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_1(0,1,0,1)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2$$

$$I_2(0,1,0,1)\big|_{(2,2,0,0)} = P_D P_F(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2$$

$$I_3(0,1,0,1)\big|_{(2,2,0,0)} = 2P_D P_F(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_1(0,1,0,0)\big|_{(2,2,0,0)} = P_F(1-P_F)(1-P_D)^2 + 2P_F(1-P_F)^3$$

$$I_2(0,1,0,0)\big|_{(2,2,0,0)} = P_D(1-P_D)(1-P_F)^2 + 2P_F(1-P_F)^3$$

$$I_3(0,1,0,0)\big|_{(2,2,0,0)} = P_D(1-P_D)(1-P_F)^2 + P_F(1-P_F)(1-P_D)^2 + P_F(1-P_F)^3$$

$$I_1(0,0,1,1)\big|_{(2,2,0,0)} = P_D^2(1-P_F)^2 + 2P_F^2(1-P_F)^2$$

$$I_2(0,0,1,1)\big|_{(2,2,0,0)} = P_F^2(1-P_D)^2 + 2P_F^2(1-P_F)^2$$

$$I_3(0,0,1,1)\big|_{(2,2,0,0)} = P_F^2(1-P_D)^2 + P_D^2(1-P_F)^2 + P_F^2(1-P_F)^2$$

$$I_1(0,0,1,0)\big|_{(2,2,0,0)} = P_D(1-P_D)(1-P_F)^2 + 2P_F(1-P_F)^3$$

$$I_2(0,0,1,0)\big|_{(2,2,0,0)} = P_F(1-P_F)(1-P_D)^2 + 2P_F(1-P_F)^3$$

$$I_3(0,0,1,0)\big|_{(2,2,0,0)} = P_F(1-P_F)(1-P_D)^2 + P_D(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3$$

$$I_1(0,0,0,1)\big|_{(2,2,0,0)} = P_D(1-P_D)(1-P_F)^2 + 2P_F(1-P_F)^3$$

$$I_2(0,0,0,1)\big|_{(2,2,0,0)} = P_F(1-P_F)(1-P_D)^2 + 2P_F(1-P_F)^3$$

$$I_3(0,0,0,1)\big|_{(2,2,0,0)} = P_F(1-P_F)(1-P_D)^2 + P_D(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3$$

$$I_1(0,0,0,0)\big|_{(2,2,0,0)} = (1-P_D)^2(1-P_F)^2 + 2(1-P_F)^4$$

$$I_2(0,0,0,0)\big|_{(2,2,0,0)} = (1-P_D)^2(1-P_F)^2 + 2(1-P_F)^4$$

$$I_3(0,0,0,0)\big|_{(2,2,0,0)} = 2(1-P_D)^2(1-P_F)^2 + (1-P_F)^4 \tag{36}$$

The conditional probabilties $p_1(y_1,y_2,y_3,y_4)\big|_{(v_1,v_2,v_3,v_4)}$ for the sensor placement $(v_1,v_2,v_3,v_4) =$

$(3, 1, 0, 0)$ are given by:

$$p_1(1,1,1,1)\big|_{(3,1,0,0)} = P_D^3 P_F \ , \ \ p_1(1,1,1,0)\big|_{(3,1,0,0)} = P_D^3(1-P_F)$$

$$p_1(1,1,0,1)\big|_{(3,1,0,0)} = P_D^2 P_F(1-P_D) \ , \ \ p_1(1,1,0,0)\big|_{(3,1,0,0)} = P_D^2(1-P_D)(1-P_F)$$

$$p_1(1,0,1,1)\big|_{(3,1,0,0)} = P_D^2 P_F(1-P_D) \ , \ \ p_1(1,0,1,0)\big|_{(3,1,0,0)} = P_D^2(1-P_D)(1-P_F)$$

$$p_1(1,0,0,1)\big|_{(3,1,0,0)} = P_D P_F(1-P_D)^2 \ , \ \ p_1(1,0,0,0)\big|_{(3,1,0,0)} = P_D(1-P_F)(1-P_D)^2$$

$$p_1(0,1,1,1)\big|_{(3,1,0,0)} = P_D^2 P_F(1-P_D) \ , \ \ p_1(0,1,1,0)\big|_{(3,1,0,0)} = P_D^2(1-P_D)(1-P_F)$$

$$p_1(0,1,0,1)\big|_{(3,1,0,0)} = P_D P_F(1-P_D)^2 \ , \ \ p_1(0,1,0,0)\big|_{(3,1,0,0)} = P_D(1-P_F)(1-P_D)^2$$

$$p_1(0,0,1,1)\big|_{(3,1,0,0)} = P_D P_F(1-P_D)^2 \ , \ \ p_1(0,0,1,0)\big|_{(3,1,0,0)} = P_D(1-P_F)(1-P_D)^2$$

$$p_1(0,0,0,1)\big|_{(3,1,0,0)} = P_F(1-P_D)^3 \ , \ \ p_1(0,0,0,0)\big|_{(3,1,0,0)} = (1-P_D)^3(1-P_F)$$

The conditional probabilties $p_2(y_1, y_2, y_3, y_4)\big|_{(v_1,v_2,v_3,v_4)}$ for the sensor placement $(v_1, v_2, v_3, v_4) = (3, 1, 0, 0)$ are given by:

$$p_2(1,1,1,1)\big|_{(3,1,0,0)} = P_D P_F^3 \ , \ \ p_2(1,1,1,0)\big|_{(3,1,0,0)} = P_F^3(1-P_D)$$

$$p_2(1,1,0,1)\big|_{(3,1,0,0)} = P_D P_F^2(1-P_F) \ , \ \ p_2(1,1,0,0)\big|_{(3,1,0,0)} = P_F^2(1-P_D)(1-P_F)$$

$$p_2(1,0,1,1)\big|_{(3,1,0,0)} = P_D P_F^2(1-P_F) \ , \ \ p_2(1,0,1,0)\big|_{(3,1,0,0)} = P_F^2(1-P_D)(1-P_F)$$

$$p_2(1,0,0,1)\big|_{(3,1,0,0)} = P_D P_F(1-P_F)^2 \ , \ \ p_2(1,0,0,0)\big|_{(3,1,0,0)} = P_F(1-P_D)(1-P_F)^2$$

$$p_2(0,1,1,1)\big|_{(3,1,0,0)} = P_D P_F^2(1-P_F) \ , \ \ p_2(0,1,1,0)\big|_{(3,1,0,0)} = P_F^2(1-P_D)(1-P_F)$$

$$p_2(0,1,0,1)\big|_{(3,1,0,0)} = P_D P_F(1-P_F)^2 \ , \ \ p_2(0,1,0,0)\big|_{(3,1,0,0)} = P_F(1-P_D)(1-P_D)^2$$

$$p_2(0,0,1,1)\big|_{(3,1,0,0)} = P_D P_F(1-P_F)^2 \ , \ \ p_2(0,0,1,0)\big|_{(3,1,0,0)} = P_F(1-P_D)(1-P_F)^2$$

$$p_2(0,0,0,1)\big|_{(3,1,0,0)} = P_D(1-P_F)^3 \ , \ \ p_2(0,0,0,0)\big|_{(3,1,0,0)} = (1-P_D)(1-P_F)^3$$

The conditional probabilties $p_3(y_1, y_2, y_3, y_4)\big|_{(v_1,v_2,v_3,v_4)}$ for the sensor placement $(v_1, v_2, v_3, v_4) =$

$(3, 1, 0, 0)$ are given by:

$$p_3(1,1,1,1)\big|_{(3,1,0,0)} = P_F^4 \ , \ p_3(1,1,1,0)\big|_{(3,1,0,0)} = P_F^3(1 - P_F)$$

$$p_3(1,1,0,1)\big|_{(3,1,0,0)} = P_F^3(1 - P_F) \ , \ p_3(1,1,0,0)\big|_{(3,1,0,0)} = P_F^2(1 - P_F)^2$$

$$p_3(1,0,1,1)\big|_{(3,1,0,0)} = P_F^3(1 - P_F) \ , \ p_3(1,0,1,0)\big|_{(3,1,0,0)} = P_F^2(1 - P_F)^2$$

$$p_3(1,0,0,1)\big|_{(3,1,0,0)} = P_F^2(1 - P_F)^2 \ , \ p_3(1,0,0,0)\big|_{(3,1,0,0)} = P_F(1 - P_F)^3$$

$$p_3(0,1,1,1)\big|_{(3,1,0,0)} = P_F^3(1 - P_F) \ , \ p_3(0,1,1,0)\big|_{(3,1,0,0)} = P_F^2(1 - P_F)^2$$

$$p_3(0,1,0,1)\big|_{(3,1,0,0)} = P_F^2(1 - P_F)^2 \ , \ p_3(0,1,0,0)\big|_{(3,1,0,0)} = P_F(1 - P_F)^3$$

$$p_3(0,0,1,1)\big|_{(3,1,0,0)} = P_F^2(1 - P_F)^2 \ , \ p_3(0,0,1,0)\big|_{(3,1,0,0)} = P_F(1 - P_F)^3$$

$$p_3(0,0,0,1)\big|_{(3,1,0,0)} = P_F(1 - P_F)^3 \ , \ p_3(0,0,0,0)\big|_{(3,1,0,0)} = (1 - P_F)^4$$

Using equation (34) and the conditional probabilities we get the following values for $I_i(y_1, y_2, y_3, y_4)\big|_{(3,1,0,0)}$:

$$I_1(1,1,1,1)\big|_{(3,1,0,0)} = P_D P_F^3 + 2P_F^4$$

$$I_2(1,1,1,1)\big|_{(3,1,0,0)} = P_D^3 P_F + 2P_F^4$$

$$I_3(1,1,1,1)\big|_{(3,1,0,0)} = P_D^3 P_F + P_D P_F^3 + P_F^4$$

$$I_1(1,1,1,0)\big|_{(3,1,0,0)} = P_F^3(1 - P_D) + 2P_F^3(1 - P_F)$$

$$I_2(1,1,1,0)\big|_{(3,1,0,0)} = P_D^3(1 - P_F) + 2P_F^3(1 - P_F)$$

$$I_3(1,1,1,0)\big|_{(3,1,0,0)} = P_D^3(1 - P_F) + P_F^3(1 - P_D) + P_F^3(1 - P_F)$$

$$I_1(1,1,0,1)\big|_{(3,1,0,0)} = P_D P_F^2(1 - P_F) + 2P_F^3(1 - P_F)$$

$$I_2(1,1,0,1)\big|_{(3,1,0,0)} = P_D^2 P_F(1 - P_D) + 2P_F^3(1 - P_F)$$

$$I_3(1,1,0,1)\big|_{(3,1,0,0)} = P_D^2 P_F(1 - P_D) + P_D P_F^2(1 - P_F) + P_F^3(1 - P_F)$$

$$I_1(1,1,0,0)\big|_{(3,1,0,0)} = P_F^2(1 - P_D)(1 - P_F) + 2P_F^2(1 - P_F)^2$$

$$I_2(1,1,0,0)\big|_{(3,1,0,0)} = P_D^2(1 - P_D)(1 - P_F) + 2P_F^2(1 - P_F)^2$$

$$I_3(1,1,0,0)\big|_{(3,1,0,0)} = P_D^2(1-P_D)(1-P_F) + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_1(1,0,1,1)\big|_{(3,1,0,0)} = P_D P_F^2(1-P_F) + 2P_F^3(1-P_F)$$

$$I_2(1,0,1,1)\big|_{(3,1,0,0)} = P_D^2 P_F(1-P_D) + 2P_F^3(1-P_F)$$

$$I_3(1,0,1,1)\big|_{(3,1,0,0)} = P_D^2 P_F(1-P_D) + P_D P_F^2(1-P_F) + P_F^3(1-P_F)$$

$$I_1(1,0,1,0)\big|_{(3,1,0,0)} = P_F^2(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2$$

$$I_2(1,0,1,0)\big|_{(3,1,0,0)} = P_D^2(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2$$

$$I_3(1,0,1,0)\big|_{(3,1,0,0)} = P_D^2(1-P_D)(1-P_F) + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_1(1,0,0,1)\big|_{(3,1,0,0)} = P_D P_F(1-P_F)^2 + 2P_F^2(1-P_F)^2$$

$$I_2(1,0,0,1)\big|_{(3,1,0,0)} = P_D P_F(1-P_D)^2 + 2P_F^2(1-P_F)^2$$

$$I_3(1,0,0,1)\big|_{(3,1,0,0)} = P_D P_F(1-P_D)^2 + P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2$$

$$I_1(1,0,0,0)\big|_{(3,1,0,0)} = P_F(1-P_D)(1-P_F)^2 + 2P_F(1-P_F)^3$$

$$I_2(1,0,0,0)\big|_{(3,1,0,0)} = P_D(1-P_F)(1-P_D)^2 + 2P_F(1-P_F)^3$$

$$I_3(1,0,0,0)\big|_{(3,1,0,0)} = P_D(1-P_F)(1-P_D)^2 + P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3$$

$$I_1(0,1,1,1)\big|_{(3,1,0,0)} = P_D P_F^2(1-P_F) + 2P_F^3(1-P_F)$$

$$I_2(0,1,1,1)\big|_{(3,1,0,0)} = P_D^2 P_F(1-P_D) + 2P_F^3(1-P_F)$$

$$I_3(0,1,1,1)\big|_{(3,1,0,0)} = P_D^2 P_F(1-P_D) + P_D P_F^2(1-P_F) + P_F^3(1-P_F)$$

$$I_1(0,1,1,0)\big|_{(3,1,0,0)} = P_F^2(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2$$

$$I_2(0,1,1,0)\big|_{(3,1,0,0)} = P_D^2(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2$$

$$I_3(0,1,1,0)\big|_{(3,1,0,0)} = P_D^2(1-P_D)(1-P_F) + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2$$

$$I_1(0,1,0,1)\big|_{(3,1,0,0)} = P_D P_F(1-P_F)^2 + 2P_F^2(1-P_F)^2$$

$$I_2(0,1,0,1)\big|_{(3,1,0,0)} = P_D P_F(1-P_D)^2 + 2P_F^2(1-P_F)^2$$

$$I_3(0,1,0,1)\big|_{(3,1,0,0)} = P_D P_F(1-P_D)^2 + P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2$$

$$I_1(0,1,0,0)\big|_{(3,1,0,0)} = P_F(1-P_D)(1-P_F)^2 + 2P_F(1-P_F)^3$$

$$I_2(0,1,0,0)\big|_{(3,1,0,0)} = P_D(1-P_F)(1-P_D)^2 + 2P_F(1-P_F)^3$$

$$I_3(0,1,0,0)\big|_{(3,1,0,0)} = P_D(1-P_F)(1-P_D)^2 + P_F(1-P_D)(1-P_F^2) + P_F(1-P_F)^3$$

$$I_1(0,0,1,1)\big|_{(3,1,0,0)} = P_D P_F (1-P_F)^2 + 2P_F^2(1-P_F)^2$$

$$I_2(0,0,1,1)\big|_{(3,1,0,0)} = P_D P_F (1-P_D)^2 + 2P_F^2(1-P_F)^2$$

$$I_3(0,0,1,1)\big|_{(3,1,0,0)} = P_D P_F (1-P_D)^2 + P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2$$

$$I_1(0,0,1,0)\big|_{(3,1,0,0)} = P_F(1-P_D)(1-P_F)^2 + 2P_F(1-P_F)^3$$

$$I_2(0,0,1,0)\big|_{(3,1,0,0)} = P_D(1-P_F)(1-P_D)^2 + 2P_F(1-P_F)^3$$

$$I_3(0,0,1,0)\big|_{(3,1,0,0)} = P_D(1-P_F)(1-P_D)^2 + P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3$$

$$I_1(0,0,0,1)\big|_{(3,1,0,0)} = P_D(1-P_F)^3 + 2P_F(1-P_F)^3$$

$$I_2(0,0,0,1)\big|_{(3,1,0,0)} = P_F(1-P_D)^3 + 2P_F(1-P_F)^3$$

$$I_3(0,0,0,1)\big|_{(3,1,0,0)} = P_D(1-P_F)^3 + P_F(1-P_D)^3 + P_F(1-P_F)^3$$

$$I_1(0,0,0,0)\big|_{(3,1,0,0)} = (1-P_D)(1-P_F)^3 + 2(1-P_F)^4$$

$$I_2(0,0,0,0)\big|_{(3,1,0,0)} = (1-P_D)^3(1-P_F) + 2(1-P_F)^4$$

$$I_3(0,0,0,0)\big|_{(3,1,0,0)} = (1-P_D)^3(1-P_F) + (1-P_D)(1-P_F)^3 + (1-P_F)^4 \qquad (37)$$

The conditional probabilties $p_1(y_1, y_2, y_3, y_4)\big|_{(v_1, v_2, v_3, v_4)}$ for the sensor placement $(v_1, v_2, v_3, v_4) = (4, 0, 0, 0)$ are given by:

$$p_1(1,1,1,1)\big|_{(4,0,0,0)} = P_D^4 \ , \ p_1(1,1,1,0)\big|_{(4,0,0,0)} = P_D^3(1-P_D)$$

$$p_1(1,1,0,1)\big|_{(4,0,0,0)} = P_D^3(1-P_D) \ , \ p_1(1,1,0,0)\big|_{(4,0,0,0)} = P_D^2(1-P_D)^2$$

$$p_1(1,0,1,1)\big|_{(4,0,0,0)} = P_D^3(1-P_D) \ , \ p_1(1,0,1,0)\big|_{(4,0,0,0)} = P_D^2(1-P_D)^2$$

$$p_1(1,0,0,1)\big|_{(4,0,0,0)} = P_D^2(1-P_D)^2 \ , \ p_1(1,0,0,0)\big|_{(4,0,0,0)} = P_D(1-P_D)^3$$

$$p_1(0,1,1,1)\big|_{(4,0,0,0)} = P_D^3(1-P_D) \ , \ p_1(0,1,1,0)\big|_{(4,0,0,0)} = P_D^2(1-P_D)^2$$

$$p_1(0,1,0,1)\big|_{(4,0,0,0)} = P_D^2(1-P_D)^2 \ , \ p_1(0,1,0,0)\big|_{(4,0,0,0)} = P_D(1-P_D)^3$$

$$p_1(0,0,1,1)\big|_{(4,0,0,0)} = P_D^2(1-P_D)^2 \ , \ p_1(0,0,1,0)\big|_{(4,0,0,0)} = P_D(1-P_D)^3$$

$$p_1(0,0,0,1)\big|_{(4,0,0,0)} = P_D(1-P_D)^3 \ , \ p_1(0,0,0,0)\big|_{(4,0,0,0)} = (1-P_D)^4$$

The conditional probabilties $p_2(y_1, y_2, y_3, y_4)\big|_{(v_1, v_2, v_3, v_4)}$ for the sensor placement $(v_1, v_2, v_3, v_4) = (4, 0, 0, 0)$ are given by:

$$p_2(1,1,1,1)\big|_{(4,0,0,0)} = P_F^4 \ , \ p_2(1,1,1,0)\big|_{(4,0,0,0)} = P_F^3(1 - P_F)$$

$$p_2(1,1,0,1)\big|_{(4,0,0,0)} = P_F^3(1 - P_F) \ , \ p_2(1,1,0,0)\big|_{(4,0,0,0)} = P_F^2(1 - P_F)^2$$

$$p_2(1,0,1,1)\big|_{(4,0,0,0)} = P_F^3(1 - P_F) \ , \ p_2(1,0,1,0)\big|_{(4,0,0,0)} = P_F^2(1 - P_F)^2$$

$$p_2(1,0,0,1)\big|_{(4,0,0,0)} = P_F^2(1 - P_F)^2 \ , \ p_2(1,0,0,0)\big|_{(4,0,0,0)} = P_F(1 - P_F)^3$$

$$p_2(0,1,1,1)\big|_{(4,0,0,0)} = P_F^3(1 - P_F) \ , \ p_2(0,1,1,0)\big|_{(4,0,0,0)} = P_F^2(1 - P_F)^2$$

$$p_2(0,1,0,1)\big|_{(4,0,0,0)} = P_F^2(1 - P_F)^2 \ , \ p_2(0,1,0,0)\big|_{(4,0,0,0)} = P_F(1 - P_F)^3$$

$$p_2(0,0,1,1)\big|_{(4,0,0,0)} = P_F^2(1 - P_F)^2 \ , \ p_2(0,0,1,0)\big|_{(4,0,0,0)} = P_F(1 - P_F)^3$$

$$p_2(0,0,0,1)\big|_{(4,0,0,0)} = P_F(1 - P_F)^3 \ , \ p_2(0,0,0,0)\big|_{(4,0,0,0)} = (1 - P_F)^4$$

It should be noted that for this placement: $p_2(y_1, y_2, y_3, y_4)\big|_{(4,0,0,0)} = p_3(y_1, y_2, y_3, y_4)\big|_{(4,0,0,0)} = p_4(y_1, y_2, y_3, y_4)\big|_{(4,0,0,0)}$. Using equation (34) and the conditional probabilities we get the following values for $I_i(y_1, y_2, y_3, y_4)\big|_{(4,0,0,0)}$:

$$I_1(1,1,1,1)\big|_{(4,0,0,0)} = 3P_F^4 \ , \ I_2(1,1,1,1)\big|_{(4,0,0,0)} = P_D^4 + 2P_F^4$$

$$I_1(1,1,1,0)\big|_{(4,0,0,0)} = 3P_F^3(1 - P_F) \ , \ I_2(1,1,1,0)\big|_{(4,0,0,0)} = P_D^3(1 - P_D) + 2P_F^3(1 - P_F)$$

$$I_1(1,1,0,1)\big|_{(4,0,0,0)} = 3P_F^3(1 - P_F) \ , \ I_2(1,1,0,1)\big|_{(4,0,0,0)} = P_D^3(1 - P_D) + 2P_F^3(1 - P_F)$$

$$I_1(1,1,0,0)\big|_{(4,0,0,0)} = 3P_F^2(1 - P_F)^2 \ , \ I_2(1,1,0,0)\big|_{(4,0,0,0)} = P_D^2(1 - P_D)^2 + 2P_F^2(1 - P_F)^2$$

$$I_1(1,0,1,1)\big|_{(4,0,0,0)} = 3P_F^3(1 - P_F) \ , \ I_2(1,0,1,1)\big|_{(4,0,0,0)} = P_D^3(1 - P_D) + 2P_F^3(1 - P_F)$$

$$I_1(1,0,1,0)\big|_{(4,0,0,0)} = 3P_F^2(1 - P_F)^2 \ , \ I_2(1,0,1,0)\big|_{(4,0,0,0)} = P_D^2(1 - P_D)^2 + 2P_F^2(1 - P_F)^2$$

$$I_1(1,0,0,1)\big|_{(4,0,0,0)} = 3P_F^2(1 - P_F)^2 \ , \ I_2(1,0,0,1)\big|_{(4,0,0,0)} = P_D^2(1 - P_D)^2 + 2P_F^2(1 - P_F)^2$$

$$I_1(1,0,0,0)\big|_{(4,0,0,0)} = 3P_F(1 - P_F)^3 \ , \ I_2(1,0,0,0)\big|_{(4,0,0,0)} = P_D(1 - P_D)^3 + 2P_F(1 - P_F)^3$$

$$I_1(0,1,1,1)\big|_{(4,0,0,0)} = 3P_F^3(1 - P_F) \ , \ I_2(0,1,1,1)\big|_{(4,0,0,0)} = P_D^3(1 - P_D) + 2P_F^3(1 - P_F)$$

$$I_1(0,1,1,0)\big|_{(4,0,0,0)} = 3P_F^2(1-P_F)^2 \ , \ I_2(0,1,1,0)\big|_{(4,0,0,0)} = P_D^2(1-P_D)^2 + 2P_F^2(1-P_F)^2$$

$$I_1(0,1,0,1)\big|_{(4,0,0,0)} = 3P_F^2(1-P_F)^2 \ , \ I_2(0,1,0,1)\big|_{(4,0,0,0)} = P_D^2(1-P_D)^2 + 2P_F^2(1-P_F)^2$$

$$I_1(0,1,0,0)\big|_{(4,0,0,0)} = 3P_F(1-P_F)^3 \ , \ I_2(0,1,0,0)\big|_{(4,0,0,0)} = P_D(1-P_D)^3 + 2P_F(1-P_F)^3$$

$$I_1(0,0,1,1)\big|_{(4,0,0,0)} = 3P_F^2(1-P_F)^2 \ , \ I_2(0,0,1,1)\big|_{(4,0,0,0)} = P_D^2(1-P_D)^2 + 2P_F^2(1-P_F)^2$$

$$I_1(0,0,1,0)\big|_{(4,0,0,0)} = 3P_F(1-P_F)^3 \ , \ I_2(0,0,1,0)\big|_{(4,0,0,0)} = P_D(1-P_D)^3 + 2P_F(1-P_F)^3$$

$$I_1(0,0,0,1)\big|_{(4,0,0,0)} = 3P_F(1-P_F)^3 \ , \ I_2(0,0,0,1)\big|_{(4,0,0,0)} = P_D(1-P_D)^3 + 2P_F(1-P_F)^3$$

$$I_1(0,0,0,0)\big|_{(4,0,0,0)} = 3(1-P_F)^4 \ , \ I_2(0,0,0,0)\big|_{(4,0,0,0)} = (1-P_D)^4 + 2(1-P_F)^4 \qquad (38)$$

Using equations (35), (36), (37), and (38) we can obtain the probability of error of each placement respectively:

$$4P_e(2,1,1,0) = \min\left\{2P_DP_F^3 + P_F^4, P_D^2P_F^2 + P_DP_F^3 + P_F^4, P_D^2P_F^2 + 2P_DP_F^3\right\} + 2\min\left\{P_DP_F^2(1-P_F)\right.$$

$$\left. + P_F^3(1-P_D) + P_F^3(1-P_F), P_D^2P_F(1-P_F) + P_F^3(1-P_D) + P_F^3(1-P_F), P_D^2P_F(1-P_F) + P_DP_F^2(1-P_F)\right.$$

$$\left. + P_F^3(1-P_F), P_D^2P_F(1-P_F) + P_DP_F^2(1-P_F) + P_F^3(1-P_D)\right\} + \min\left\{2P_F^2(1-P_D)(1-P_F) + \right.$$

$$\left. P_F^2(1-P_F)^2, P_D^2(1-P_F)^2 + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2, P_D^2(1-P_F)^2 + 2P_F^2(1-P_D)(1-P_F)\right\} +$$

$$2\min\left\{2P_DP_F^2(1-P_F) + P_F^3(1-P_F), P_DP_F^2(1-P_D) + P_DP_F^2(1-P_F) + P_F^3(1-P_F), P_DP_F^2(1-P_D)\right.$$

$$\left. + 2P_DP_F^2(1-P_F)\right\} + 4\min\left\{P_DP_F(1-P_F)^2 + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2, P_DP_F(1-P_D)(1-P_F)\right.$$

$$\left. + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2, P_DP_F(1-P_D)(1-P_F) + P_DP_F(1-P_F)^2 + P_F^2(1-P_F)^2,\right.$$

$$\left. P_DP_F(1-P_D)(1-P_F) + P_DP_F(1-P_F)^2 + P_F^2(1-P_D)(1-P_F)\right\} + 2\min\left\{2P_F(1-P_D)(1-P_F)^2\right.$$

$$\left. + P_F(1-P_F)^3, P_D(1-P_D)(1-P_F)^2 + P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3, P_D(1-P_D)(1-P_F)^2 + \right.$$

$$\left. 2P_F(1-P_D)(1-P_F)^2\right\} + \min\left\{2P_DP_F(1-P_F)^2 + P_F^2(1-P_F)^2, P_F^2(1-P_D)^2 + P_DP_F(1-P_F)^2\right.$$

$$\left. + P_F^2(1-P_F)^2, P_F^2(1-P_D)^2 + 2P_DP_F(1-P_F)^2\right\} + 2\min\left\{P_D(1-P_F)^3 + P_F(1-P_D)(1-P_F)^2\right.$$

$$P_F(1-P_F)^3, P_F(1-P_F)(1-P_D)^2 + P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3, P_F(1-P_F)(1-P_D)^2$$

$$+ P_D(1-P_F)^3 + P_F(1-P_F)^3, P_F(1-P_F)(1-P_D)^2 + P_D(1-P_F)^3 + P_F(1-P_D)(1-P_F)^2 \Big\} +$$

$$\min\Big\{2(1-P_D)(1-P_F)^3 + (1-P_F)^4, (1-P_D)^2(1-P_F)^2 + (1-P_D)(1-P_F)^3 + (1-P_F)^4,$$

$$(1-P_D)^2(1-P_F)^2 + 2(1-P_D)(1-P_F)^3\Big\} \tag{39}$$

Using the fact $P_D \geq P_F$, the above expression can be simplified to give:

$$4P_e(2,1,1,0) = 2P_D P_F^3 + P_F^4 + 2P_D P_F^2(1-P_F) + 2P_F^3(1-P_D) + 2P_F^3(1-P_F) + 2P_F^2(1-P_D)(1-P_F)$$

$$+ P_F^2(1-P_F)^2 + 2P_D P_F^2(1-P_D) + 2P_D P_F^2(1-P_F) + 2P_F^3(1-P_F) + 4P_D P_F(1-P_D)(1-P_F) +$$

$$4P_F^2(1-P_D)(1-P_F) + 4P_F^2(1-P_F)^2 + 2\min\{2P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3, P_D(1-P_D)(1-P_F)^2$$

$$+ 2P_F(1-P_D)(1-P_F)^2\} + P_F^2(1-P_D)^2 + P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2 + 2P_F(1-P_F)(1-P_D)^2$$

$$+ 2P_F(1-P_D)(1-P_F)^2 + 2P_F(1-P_F)^3 + (1-P_D)^2(1-P_F)^2 + 2(1-P_D)(1-P_F)^3 \tag{40}$$

There are two cases here, $P_D(1-P_D) \geq P_F(1-P_F)$ and $P_D(1-P_D) < P_F(1-P_F)$:

Case 1: $\mathbf{P_D(1-P_D) \geq P_F(1-P_F)}$

$$4P_e(2,1,1,0) = 2P_D P_F^3 + P_F^4 + 2P_D P_F^2(1-P_F) + 2P_F^3(1-P_D) + 2P_F^3(1-P_F) + 2P_F^2(1-P_D)(1-P_F)$$

$$+ P_F^2(1-P_F)^2 + 2P_D P_F^2(1-P_D) + 2P_D P_F^2(1-P_F) + 2P_F^3(1-P_F) + 4P_D P_F(1-P_D)(1-P_F) +$$

$$4P_F^2(1-P_D)(1-P_F) + 4P_F^2(1-P_F)^2 + 4P_F(1-P_D)(1-P_F)^2 + 2P_F(1-P_F)^3 + P_F^2(1-P_D)^2$$

$$+ P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2 + 2P_F(1-P_F)(1-P_D)^2 + 2P_F(1-P_D)(1-P_F)^2 + 2P_F(1-P_F)^3$$

$$+ (1-P_D)^2(1-P_F)^2 + 2(1-P_D)(1-P_F)^3 \tag{41}$$

Case 2: $\mathbf{P_D(1-P_D) < P_F(1-P_F)}$

$$4P_e(2,1,1,0) = 2P_D P_F^3 + P_F^4 + 2P_D P_F^2(1-P_F) + 2P_F^3(1-P_D) + 2P_F^3(1-P_F) + 2P_F^2(1-P_D)(1-P_F)$$

$$+ P_F^2(1-P_F)^2 + 2P_D P_F^2(1-P_D) + 2P_D P_F^2(1-P_F) + 2P_F^3(1-P_F) + 4P_D P_F(1-P_D)(1-P_F) +$$

$$4P_F^2(1-P_D)(1-P_F) + 4P_F^2(1-P_F)^2 + 4P_F(1-P_D)(1-P_F)^2 + 2P_D(1-P_D)(1-P_F)^2 + P_F^2(1-P_D)^2$$

$$+ P_D P_F(1-P_F)^2 + P_F^2(1-P_F)^2 + 2P_F(1-P_F)(1-P_D)^2 + 2P_F(1-P_D)(1-P_F)^2 + 2P_F(1-P_F)^3$$

$$+ (1-P_D)^2(1-P_F)^2 + 2(1-P_D)(1-P_F)^3 \tag{42}$$

Now we provide an expression for $P_e(2,2,0,0)$ using (5), (34), and (36):

$$4P_e(2,2,0,0) = \min\{P_D^2 P_F^2 + 2P_F^4, 2P_D^2 P_F^2 + P_F^4\} + 4\min\{P_D P_F^2(1-P_D) + 2P_F^3(1-P_F), P_D^2 P_F(1-P_F) +$$

$$2P_F^3(1-P_F), P_D^2 P_F(1-P_F) + P_D P_F^2(1-P_D) + P_F^3(1-P_F)\} + 2\min\{P_F^2(1-P_D)^2 + 2P_F^2(1-P_F)^2, P_D^2(1-P_F)^2 +$$

$$2P_F^2(1-P_F)^2, P_D^2(1-P_F)^2 + P_F^2(1-P_D)^2 + P_F^2(1-P_F)^2\} + 4\min\{P_D P_F(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2,$$

$$2P_D P_F(1-P_D)(1-P_F) + P_F^2(1-P_F)^2\} + 4\min\{P_D(1-P_D)(1-P_F)^2 + 2P_F(1-P_F)^3, P_F(1-P_F)(1-P_D)^2 +$$

$$2P_F(1-P_F)^3, P_F(1-P_F)(1-P_D)^2 + P_D(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3\} + \min\{(1-P_D)^2(1-P_F)^2 + 2(1-P_F)^4,$$

$$2(1-P_D)^2(1-P_F)^2 + (1-P_F)^4\} \tag{43}$$

Using the fact $P_D \geq P_F$, the above expression can be simplified to give:

$$4P_e(2,2,0,0) = P_D^2 P_F^2 + 2P_F^4 + 4P_D P_F^2(1-P_D) + 8P_F^3(1-P_F) + 2P_F^2(1-P_D)^2 + 4P_F^2(1-P_F)^2 +$$

$$4\min\{P_D P_F(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2, 2P_D P_F(1-P_D)(1-P_F) + P_F^2(1-P_F)^2\} +$$

$$4\min\{P_F(1-P_F)(1-P_D)^2 + 2P_F(1-P_F)^3, P_F(1-P_F)(1-P_D)^2 + P_D(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3\}$$

$$+ 2(1-P_D)^2(1-P_F)^2 + (1-P_F)^4 \tag{44}$$

Now there are two cases here: $P_D(1-P_D) \geq P_F(1-P_F)$ and $P_D(1-P_D) < P_F(1-P_F)$

Case 1: $\mathbf{P_D(1-P_D) \geq P_F(1-P_F)}$

$$4P_e(2,2,0,0) = P_D^2 P_F^2 + 2P_F^4 + 4P_D P_F^2(1-P_D) + 8P_F^3(1-P_F) + 2P_F^2(1-P_D)^2 + 4P_F^2(1-P_F)^2 +$$

$$4P_D P_F(1-P_D)(1-P_F) + 8P_F^2(1-P_F)^2 + 4P_F(1-P_F)(1-P_D)^2 + 8P_F(1-P_F)^3 + 2(1-P_D)(1-P_F)^2$$

$$+ \ (1-P_F)^4 \tag{45}$$

Case 2: $P_D(1-P_D) < P_F(1-P_F)$

$$4P_e(2,2,0,0) = P_D^2P_F^2 + 2P_F^4 + 4P_DP_F^2(1-P_D) + 8P_F^3(1-P_F) + 2P_F^2(1-P_D)^2 + 4P_F^2(1-P_F)^2 +$$

$$8P_DP_F(1-P_D)(1-P_F) + 4P_F^2(1-P_F)^2 + 4P_F(1-P_F)(1-P_D)^2 + 4P_D(1-P_D)(1-P_F)^2 + 4P_F(1-P_F)^3$$

$$+ \ 2(1-P_D)^2(1-P_F)^2 + (1-P_F)^4 \tag{46}$$

Now we provide an expression for $P_e(3,1,0,0)$ using (5), (34), and (37):

$$4P_e(3,1,0,0) = \min\{P_DP_F^3 + 2P_F^4, P_D^3P_F + 2P_F^4, P_D^3P_F + P_DP_F^3 + P_F^4\} + \min\{P_F^3(1-P_D) + 2P_F^3(1-P_F),$$

$$P_D^3(1-P_F) + 2P_F^3(1-P_F), P_D^3(1-P_F) + P_F^3(1-P_D) + P_F^3(1-P_F)\} + 3\min\{P_DP_F^2(1-P_F) + 2P_F^3(1-P_F),$$

$$P_D^2P_F(1-P_D) + 2P_F^3(1-P_F), P_D^2P_F(1-P_D) + P_DP_F^2(1-P_F) + P_F^3(1-P_F)\} + 3\min\{P_F^2(1-P_D)(1-P_F)$$

$$+ 2P_F^2(1-P_F)^2, P_D^2(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2, P_D^2(1-P_D)(1-P_F) + P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_F)^2\}$$

$$+ 3\min\{P_DP_F(1-P_F)^2 + 2P_F^2(1-P_F)^2, P_DP_F(1-P_D)^2 + 2P_F^2(1-P_F)^2, P_DP_F(1-P_D)^2 + P_DP_F(1-P_F)^2 +$$

$$P_F^2(1-P_F)^2\} + 3\min\{P_F(1-P_D)(1-P_F)^2 + 2P_F(1-P_F)^3, P_D(1-P_F)(1-P_D)^2 + 2P_F(1-P_F)^3,$$

$$P_D(1-P_F)(1-P_D)^2 + P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3\} + \min\{P_D(1-P_F)^3 + 2P_F(1-P_F)^3, P_F(1-P_D)^3 +$$

$$2P_F(1-P_F)^3, P_F(1-P_D)^3 + P_D(1-P_F)^3 + P_F(1-P_F)^3\} + \min\{(1-P_D)(1-P_F)^3 + 2(1-P_F)^4, (1-P_D)^3(1-P_F)$$

$$+ 2(1-P_F)^4, (1-P_D)^3(1-P_F) + (1-P_D)(1-P_F)^3 + (1-P_F)^4\} \tag{47}$$

Using the fact $P_D \geq P_F$, the above expression can be simplified to give:

$$4P_e(3,1,0,0) = P_DP_F^3 + 2P_F^4 + P_F^3(1-P_D) + 2P_F^3(1-P_F) + 3P_DP_F\min\{P_D(1-P_D), P_F(1-P_F)\}$$

$$+ 6P_F^3(1-P_F) + 3\min\{P_F^2(1-P_D)(1-P_F) + 2P_F^2(1-P_F)^2, P_D^2(1-P_D)(1-P_F) + P_F^2(1-P_D)(1-P_F)$$

$$P_F^2(1-P_F)^2\} + 3P_DP_F(1-P_D)^2 + 6P_F^2(1-P_F)^2 + 3\min\{P_F(1-P_D)(1-P_F)^2 + 2P_F(1-P_F)^3,$$

$$P_D(1-P_F)(1-P_D)^2 + P_F(1-P_D)(1-P_F)^2 + P_F(1-P_F)^3\} + P_F(1-P_D)^3 + 2P_F(1-P_F)^3$$

$$+ (1-P_D)^3(1-P_F) + (1-P_F)^4 \tag{48}$$

We have four cases here which are given below:

Case W: $P_D(1-P_D) \geq P_F(1-P_F), P_D^2(1-P_D) \geq P_F^2(1-P_F), P_D(1-P_D)^2 \geq P_F(1-P_F)^2$

$$4P_e(3,1,0,0) = P_D P_F^3 + 2P_F^4 + P_F^3(1-P_D) + 2P_F^3(1-P_F) + 3P_D P_F^2(1-P_F) + 6P_F^3(1-P_F)$$

$$3P_F^2(1-P_D)(1-P_F) + 6P_F^2(1-P_F)^2 + 3P_D P_F(1-P_D)^2 + 6P_F^2(1-P_F)^2 + 3P_F(1-P_D)(1-P_F)^2$$

$$6P_F(1-P_F)^3 + P_F(1-P_D)^3 + 2P_F(1-P_F)^3 + (1-P_D)^3(1-P_F) + (1-P_D)(1-P_F)^3 + (1-P_F)^4 \tag{49}$$

Case X: $P_D(1-P_D) \geq P_F(1-P_F), P_D^2(1-P_D) \geq P_F^2(1-P_F), P_D(1-P_D)^2 < P_F(1-P_F)^2$

$$4P_e(3,1,0,0) = P_D P_F^3 + 2P_F^4 + P_F^3(1-P_D) + 2P_F^3(1-P_F) + 3P_D P_F^2(1-P_F) + 6P_F^3(1-P_F)$$

$$+3P_F^2(1-P_D)(1-P_F) + 6P_F^2(1-P_F)^2 + 3P_D P_F(1-P_D)^2 + 6P_F^2(1-P_F)^2 + 3P_D(1-P_F)(1-P_D)^2 +$$

$$3P_F(1-P_D)(1-P_F)^2 + 3P_F(1-P_F)^3 + P_F(1-P_D)^3 + 2P_F(1-P_F)^3 + (1-P_D)^3(1-P_F)$$

$$+ (1-P_D)(1-P_F)^3 + (1-P_F)^4 \tag{50}$$

Case Y: $P_D(1-P_D) < P_F(1-P_F), P_D^2(1-P_D) \geq P_F^2(1-P_F), P_D(1-P_D)^2 < P_F(1-P_F)^2$

$$4P_e(3,1,0,0) = P_D P_F^3 + 2P_F^4 + P_F^3(1-P_D) + 2P_F^3(1-P_F) + 3P_D^2 P_F(1-P_D) + 6P_F^3(1-P_F)$$

$$+3P_F^2(1-P_D)(1-P_F) + 6P_F^2(1-P_F)^2 + 3P_D P_F(1-P_D)^2 + 6P_F^2(1-P_F)^2 + 3P_D(1-P_F)(1-P_D)^2$$

$$3P_F(1-P_D)(1-P_F)^2 + 3P_F(1-P_F)^3 + P_F(1-P_D)^3 + 2P_F(1-P_F)^3 + (1-P_D)^3(1-P_F)$$

$$+ (1-P_D)(1-P_F)^3 + (1-P_F)^4 \tag{51}$$

Case Z: $P_D(1-P_D) < P_F(1-P_F), P_D^2(1-P_D) < P_F^2(1-P_F), P_D(1-P_D)^2 < P_F(1-P_F)^2$

$$4P_e(3,1,0,0) = P_D P_F^3 + 2P_F^4 + P_F^3(1-P_D) + 2P_F^3(1-P_F) + 3P_D^2 P_F(1-P_D) + 6P_F^3(1-P_F)$$

$$3P_D^2(1-P_D)(1-P_F) + 3P_F^2(1-P_D)(1-P_F) + 3P_F^2(1-P_F)^2 + 3P_D P_F(1-P_D)^2 + 6P_F^2(1-P_F)^2$$

$$3P_D(1-P_F)(1-P_D)^2 + 3P_F(1-P_D)(1-P_F)^2 + 3P_F(1-P_F)^3 + P_F(1-P_D)^3 + 2P_F(1-P_F)^3$$

$$+ (1-P_D)^3(1-P_F) + (1-P_D)(1-P_F)^3 + (1-P_F)^4 \tag{52}$$

Now we provide an expression for $P_e(4,0,0,0)$ using (5), (34), and (38):

$$4P_e(4,0) = \min\{3P_F^4, P_D^4 + 2P_F^4\} + 4\min\{3P_F^3(1-P_F), P_D^3(1-P_D) + 2P_F^3(1-P_F)\}$$

$$+ 6\min\{3P_F^2(1-P_F)^2, P_D^2(1-P_D)^2 + 2P_F^2(1-P_F)^2\} + 4\min\{3P_F(1-P_F)^3, P_D(1-P_D)^3$$

$$+ 2P_F(1-P_F)^3\} + \min\{3(1-P_F)^4, (1-P_D)^4 + 2(1-P_F)^4\} \tag{53}$$

Using the fact $P_D \geq P_F$, the above expression can be simplified to give:

$$4P_e(4,0) = 3P_F^4 + 4\min\{3P_F^3(1-P_F), P_D^3(1-P_D) + 2P_F^3(1-P_F)\}$$

$$+ 6\min\{3P_F^2(1-P_F)^2, P_D^2(1-P_D)^2 + 2P_F^2(1-P_F)^2\} + 4\min\{3P_F(1-P_F)^3, P_D(1-P_D)^3$$

$$+ 2P_F(1-P_F)^3\} + (1-P_D)^4 + 2(1-P_F)^4 \tag{54}$$

We have four cases here which are given as follows:

Case A: $\mathbf{P_D^3(1-P_D) \geq P_F^3(1-P_F), P_D^2(1-P_D)^2 \geq P_F^2(1-P_F)^2, P_D(1-P_D)^3 \geq P_F(1-P_F)^3}$

$$4P_e(4,0,0,0) = 3P_F^4 + 12P_F^3(1-P_F) + 18P_F^2(1-P_F)^2 + 12P_F(1-P_F)^3 + (1-P_D)^4 + 2(1-P_F)^4 \tag{55}$$

Case B: $\mathbf{P_D^3(1-P_D) \geq P_F^3(1-P_F), P_D^2(1-P_D)^2 \geq P_F^2(1-P_F)^2, P_D(1-P_D)^3 < P_F(1-P_F)^3}$

$$4P_e(4,0,0,0) = 3P_F^4 + 12P_F^3(1-P_F) + 18P_F^2(1-P_F)^2 + 4P_D(1-P_D)^3 + 8P_F(1-P_F)^3$$

$$+ (1-P_D)^4 + 2(1-P_F)^4 \tag{56}$$

Case C: $\mathbf{P_D^3(1-P_D) \geq P_F^3(1-P_F), P_D^2(1-P_D)^2 < P_F^2(1-P_F)^2, P_D(1-P_D)^3 < P_F(1-P_F)^3}$

$$4P_e(4,0,0,0) = 3P_F^4 + 12P_F^3(1-P_F) + 6P_D^2(1-P_D)^2 + 12P_F^2(1-P_F)^2 + 4P_D(1-P_D)^3$$

$$+ 8P_F(1 - P_F)^3 + (1 - P_D)^4 + 2(1 - P_F)^4 \tag{57}$$

Case C: $P_D^3(1 - P_D) < P_F^3(1 - P_F), P_D^2(1 - P_D)^2 < P_F^2(1 - P_F)^2, P_D(1 - P_D)^3 < P_F(1 - P_F)^3$

$$4P_e(4, 0, 0, 0) = 3P_F^4 + 4P_D^3(1 - P_D) + 8P_F^3(1 - P_F) + 6P_D^2(1 - P_D)^2 + 12P_F^2(1 - P_F)^2 +$$

$$4P_D(1 - P_D)^3 + 8P_F(1 - P_F)^3 + (1 - P_D)^4 + 2(1 - P_F)^4 \tag{58}$$

Combining the aforementioned cases for equations (39), (43), (47), and (53) we obtain the following distinct cases:

Case AW1: $P_D^3(1 - P_D) \geq P_F^3(1 - P_F), P_D^2(1 - P_D)^2 \geq P_F^2(1 - P_F)^2, P_D(1 - P_D)^3 \geq P_F(1 - P_F)^3, P_D(1 - P_D) \geq P_F(1 - P_F), P_D^2(1 - P_D) \geq P_F^2(1 - P_F), P_D(1 - P_D)^2 \geq P_F(1 - P_F)^2$

Case BW1: $P_D^3(1 - P_D) \geq P_F^3(1 - P_F), P_D^2(1 - P_D)^2 \geq P_F^2(1 - P_F)^2, P_D(1 - P_D)^3 < P_F(1 - P_F)^3, P_D(1 - P_D) \geq P_F(1 - P_F), P_D^2(1 - P_D) \geq P_F^2(1 - P_F), P_D(1 - P_D)^2 \geq P_F(1 - P_F)^2$

Case BX1: $P_D^3(1 - P_D) \geq P_F^3(1 - P_F), P_D^2(1 - P_D)^2 \geq P_F^2(1 - P_F)^2, P_D(1 - P_D)^3 < P_F(1 - P_F)^3, P_D(1 - P_D) \geq P_F(1 - P_F), P_D^2(1 - P_D) \geq P_F^2(1 - P_F), P_D(1 - P_D)^2 < P_F(1 - P_F)^2$

Case CX1: $P_D^3(1 - P_D) \geq P_F^3(1 - P_F), P_D^2(1 - P_D)^2 < P_F^2(1 - P_F)^2, P_D(1 - P_D)^3 < P_F(1 - P_F)^3, P_D(1 - P_D) \geq P_F(1 - P_F), P_D^2(1 - P_D) \geq P_F^2(1 - P_F), P_D(1 - P_D)^2 < P_F(1 - P_F)^2$

Case CY2: $P_D^3(1 - P_D) \geq P_F^3(1 - P_F), P_D^2(1 - P_D)^2 < P_F^2(1 - P_F)^2, P_D(1 - P_D)^3 < P_F(1 - P_F)^3, P_D(1 - P_D) < P_F(1 - P_F), P_D^2(1 - P_D) \geq P_F^2(1 - P_F), P_D(1 - P_D)^2 < P_F(1 - P_F)^2$

Case CZ2: $P_D^3(1 - P_D) \geq P_F^3(1 - P_F), P_D^2(1 - P_D)^2 < P_F^2(1 - P_F)^2, P_D(1 - P_D)^3 < P_F(1 - P_F)^3, P_D(1 - P_D) < P_F(1 - P_F), P_D^2(1 - P_D) < P_F^2(1 - P_F), P_D(1 - P_D)^2 < P_F(1 - P_F)^2$

Case DZ2: $P_D^3(1 - P_D) < P_F^3(1 - P_F), P_D^2(1 - P_D)^2 < P_F^2(1 - P_F)^2, P_D(1 - P_D)^3 < P_F(1 - P_F)^3, P_D(1 - P_D) < P_F(1 - P_F), P_D^2(1 - P_D) < P_F^2(1 - P_F), P_D(1 - P_D)^2 < P_F(1 - P_F)^2$

We will analyze each of these seven cases in detail and determine their respective optimal placements.

Case AW1: $P_D^3(1-P_D) \geq P_F^3(1-P_F), P_D^2(1-P_D)^2 \geq P_F^2(1-P_F)^2, P_D(1-P_D)^3 \geq P_F(1-P_F)^3, P_D(1-P_D) \geq P_F(1-P_F), P_D^2(1-P_D) \geq P_F^2(1-P_F), P_D(1-P_D)^2 \geq P_F(1-P_F)^2$

For this case the probability of error for the placements are given by equations (41), (45), (49), and (55) respectively. For this case we will show that $P_e(2,1,1,0) \leq P_e(2,2,0,0), P_e(3,1,0,0), P_e(4,0,0,0)$. Now using equations (41) and (45) we can write:

$$4P_e(2,2,0,0) - 4P_e(2,1,1,0) = P_D^2 P_F^2 + P_F^4 + 2P_D P_F^2(1-P_D) + 4P_F^3(1-P_F) + P_F^2(1-P_D)^2$$

$$+6P_F^2(1-P_F)^2 + 2P_F(1-P_F)(1-P_D)^2 + 4P_F(1-P_F)^3 + (1-P_D)^2(1-P_F)^2 + (1-P_F)^4 - 2P_D P_F^3$$

$$- 2P_F^3(1-P_D) - 4P_D P_F^2(1-P_F) - 6P_F^2(1-P_D)(1-P_F) - 6P_F(1-P_D)(1-P_F)^2 -$$

$$P_D P_F(1-P_F)^2 - 2(1-P_D)(1-P_F)^3$$

Simplifying this equation we get:

$$4P_e(2,2,0,0) - 4P_e(2,1,1,0) = P_F^2(P_D-P_F)^2 + P_F(P_D-P_F)\{1+2(P_D-P_F)\} + (1-P_F)^2(P_D-P_F)^2$$

$$+ P_F^2(1-P_D)(1-P_F) + P_F^2(1-P_D)^2 \geq 0$$

$$\Rightarrow P_e(2,2,0,0) \geq P_e(2,1,1,0)$$

Using equations (41) and (49) we can write:

$$4P_e(3,1,0,0) - 4P_e(2,1,1,0) = P_D P_F^3 + 2P_F^4 + P_F^3(1-P_D) + 2P_F^3(1-P_F) + 3P_D P_F^2(1-P_F)$$

$$+6P_F^3(1-P_F) + 3P_F^2(1-P_D)(1-P_F) + 12P_F^2(1-P_F)^2 + 3P_D P_F(1-P_D)^2 + 3P_F(1-P_D)(1-P_F)^2$$

$$8P_F(1-P_F)^3 + P_F(1-P_D)^3 + (1-P_D)^3(1-P_F) + (1-P_D)(1-P_F)^3 + (1-P_F)^4 - 2P_D P_F^3 - P_F^4$$

$$-2P_D P_F^2(1-P_F) - 2P_F^3(1-P_D) - 2P_F^3(1-P_F) - 2P_F^2(1-P_D)(1-P_F) - P_F^2(1-P_F)^2 - 2P_D P_F^2(1-P_D)$$

$$-2P_D P_F^2(1-P_F) - 2P_F^3(1-P_F) - 4P_D P_F(1-P_D)(1-P_F) - 4P_F^2(1-P_D)(1-P_F) - 4P_F^2(1-P_F)^2$$

$$- 4P_F(1-P_D)(1-P_F)^2 - 2P_F(1-P_F)^3 - P_F^2(1-P_D)^2 - P_D P_F(1-P_F)^2 - P_F^2(1-P_F)^2$$

$$-2P_F(1-P_F)(1-P_D)^2 - 2P_F(1-P_D)(1-P_F)^2 - 2P_F(1-P_F)^3 - (1-P_D)^2(1-P_F)^2 - 2(1-P_D)(1-P_F)^3$$

Simplifying this expression we get:

$$4P_e(3,1,0,0) - 4P_e(2,1,1,0) = P_D(P_D - P_F)\{1 - (P_D + P_F) + (1 - 2P_F^2)\} + 2P_F^4$$

Using the fact that $P_a(1 - P_a) \geq P_b(1 - P_b)$ we get:

$$P_e(3,1,0,0) \geq P_e(2,1,1,0)$$

Using equations (41) and (55) we can write:

$$4P_e(4,0,0,0) - 4P_e(2,1,10) = 3P_F^4 + 12P_F^3(1-P_F) + 18P_F^2(1-P_F)^2 + 12P_F(1-P_F)^3 + (1-P_D)^4 + 2(1-P_F)^4$$

$$-2P_DP_F^3 - P_F^4 - 2P_DP_F^2(1-P_F) - 2P_F^3(1-P_D) - 2P_F^3(1-P_F) - 2P_F^2(1-P_D)(1-P_F) - P_F^2(1-P_F)^2$$

$$-2P_DP_F^2(1-P_D) - 2P_DP_F^2(1-P_F) - 2P_F^3(1-P_F) - 4P_DP_F(1-P_D)(1-P_F) - 4P_F^2(1-P_D)(1-P_F) -$$

$$4P_F^2(1-P_F)^2 - 4P_F(1-P_D)(1-P_F)^2 - 2P_F(1-P_F)^3 - P_F^2(1-P_D)^2 - P_DP_F(1-P_F)^2 - P_F^2(1-P_F)^2$$

$$-2P_F(1-P_F)(1-P_D)^2 - 2P_F(1-P_D)(1-P_F)^2 - 2P_F(1-P_F)^3 - (1-P_D)^2(1-P_F)^2 - 2(1-P_D)(1-P_F)^3$$

Simplifying the above expression we get:

$$4P_e(4,0,0,0) - 4P_e(2,1,1,0) = P_D(P_D - P_F)(5 - 4P_D) + (P_D^4 - P_F^4) + 2P_D^2(P_D^2 - P_F^2) \geq 0$$

$$\Rightarrow P_e(4,0,0,0) \geq P_e(2,1,1,0)$$

This proves that for this case:

$$P_e(2,1,1,0) \leq \min\{P_e(2,2,0,0), P_e(3,1,0,0), P_e(4,0,0,0)\}$$

Case BW1: $P_D^3(1-P_D) \geq P_F^3(1-P_F), P_D^2(1-P_D)^2 \geq P_F^2(1-P_F)^2, P_D(1-P_D)^3 < P_F(1-P_F)^3, P_D^2(1-P_D) \geq P_F^2(1-P_F), P_D(1-P_D)^2 \geq P_F(1-P_F)^2, P_D(1-P_D) \geq P_F(1-P_F)$

For this case the probability of error for the placements are given by equations (41), (45), (49), and (56) respectively. Using the results of case AW1 we can conclude that $P_e(2,1,1,0) \leq P_e(2,2,0,0), P_e(3,1,0,0)$. We need to show that $P_e(2,1,1,0) \leq P_e(4,0,0,0)$. Using equations (41) and (56) we can write:

$$4P_e(4,0,0,0)-4P_e(2,1,1,0) = 3P_F^4+12P_F^3(1-P_F)+18P_F^2(1-P_F)+4P_D(1-P_D)^3+8P_F(1-P_F)^3+$$

$$(1-P_D)^4+2(1-P_F)^4-2P_DP_F^3-P_F^4-2P_DP_F^2(1-P_F)-2P_F^3(1-P_D)-2P_F^3(1-P_F)-2P_F^2(1-P_D)(1-P_F)$$

$$-P_F^2(1-P_F)^2-2P_DP_F^2(1-P_D)-2P_DP_F^2(1-P_F)-2P_F^3(1-P_F)-4P_DP_F(1-P_D)(1-P_F)$$

$$-4P_F^2(1-P_D)(1-P_F)-4P_F^2(1-P_F)^2-4P_F(1-P_D)(1-P_F)^2-2P_F(1-P_F)^3-P_F^2(1-P_D)^2$$

$$-P_DP_F(1-P_F)^2-P_F^2(1-P_F)^2-2P_F(1-P_F)(1-P_D)^2-2P_F(1-P_D)(1-P_F)^2-2P_F(1-P_F)^3$$

$$- \quad (1-P_D)^2(1-P_F)^2-2(1-P_D)(1-P_F)^3$$

Simplifying this expression we get:

$$4P_e(4,0,0,0)-4P_e(2,1,1,0) = (P_a-P_b)\Big\{4-12P_b(1-P_b)-7P_a(1-P_a)+P_a^2+12P_aP_b$$

$$-3P_a^3-5P_aP_b^2-3P_a^2P_b-4P_b^3\Big\} \geq 0$$

$$\Rightarrow \quad P_e(4,0,0,0) \geq P_e(2,1,1,0)$$

Therefore, we can conclude for this case:

$$P_e(2,1,1,0) \leq \min\{P_e(2,2,0,0), P_e(3,1,0,0), P_e(4,0,0,0)\}$$

Case BX1: $P_D^3(1-P_D) \geq P_F^3(1-P_F), P_D^2(1-P_D)^2 \geq P_F^2(1-P_F)^2, P_D(1-P_D)^3 < P_F(1-P_F)^3, P_D^2(1-P_D) \geq P_F^2(1-P_F), P_D(1-P_D)^2 < P_F(1-P_F)^2, P_D(1-P_D) \geq P_F(1-P_F)$

For this case the probability of error for the placements are given by equations (41), (45), (50), and (56) respectively. Using the results of case BW1 we can conclude that $P_e(2,1,1,0) \leq P_e(2,2,0,0), P_e(4,0,0,0)$. We will show that $P_e(3,1,0,0) \geq P_e(2,1,1,0)$. Using equations (41) and (50) we can write:

$$4P_e(3,1,0,0) - 4P_e(2,1,1,0) = P_D P_F^3 + 2P_F^4 + P_F^3(1-P_D) + 2P_F^3(1-P_F) + 3P_D P_F^2(1-P_F)+$$

$$+6P_F^3(1-P_F)+3P_F^2(1-P_D)(1-P_F)+12P_F^2(1-P_F)^2+3P_D P_F(1-P_D)^2+3P_D(1-P_F)(1-P_D)^2+$$

$$3P_F(1-P_D)(1-P_F)^2+3P_F(1-P_F)^3+P_F(1-P_D)^3+2P_F(1-P_F)^3+(1-P_D)^3(1-P_F)+(1-P_D)(1-P_F)^3$$

$$+(1-P_F)^4-2P_D P_F^3-P_F^4-2P_D P_F^2(1-P_F)-2P_F^3(1-P_D)-2P_F^3(1-P_F)-2P_F^2(1-P_D)(1-P_F)$$

$$-P_F^2(1-P_F)^2-2P_D P_F^2(1-P_D)-2P_D P_F^2(1-P_F)-2P_F^3(1-P_F)-4P_D P_F(1-P_D)(1-P_F)$$

$$-4P_F^2(1-P_D)(1-P_F)-4P_F^2(1-P_F)^2-4P_F(1-P_D)(1-P_F)^2-2P_F(1-P_F)^3-P_F^2(1-P_D)^2$$

$$-P_D P_F(1-P_F)^2-P_F^2(1-P_F)^2-2P_F(1-P_F)(1-P_D)^2-2P_F(1-P_D)(1-P_F)^2-2P_F(1-P_F)^3$$

$$-(1-P_D)^2(1-P_F)^2-2(1-P_D)(1-P_F)^3$$

Simplifying the above expression we get:

$$4P_e(3,1,0,0)-4P_e(2,1,1,0) = (1-P_D)(3P_D-2P_D P_F-P_D^2)+P_D(P_D^2-P_F)^3+(1-P_F)(9P_F-3P_D P_F+2P_D^2 P_F)$$

$$+3P_F^4+3P_D P_F^2 \geq 0$$

This implies that:

$$P_e(2,1,1,0) \leq \min\{P_e(2,2,0,0), P_e(3,1,0,0), P_e(4,0,0,0)\}$$

Case CX1: $P_D^3(1-P_D) \geq P_F^3(1-P_F), P_D^2(1-P_D)^2 < P_F^2(1-P_F)^2, P_D(1-P_D)^3 < P_F(1-P_F)^3, P_D^2(1-P_D) \geq P_F^2(1-P_F), P_D(1-P_D)^2 < P_F(1-P_F)^2, P_D(1-P_D) \geq P_F(1-P_F)$

For this case the probability of error for the placements are given by equations (41), (45), (50), and (57) respectively. From case BX1 we get $P_e(2,1,1,0) \leq P_e(2,2,0,0), P_e(3,1,0,0)$. We will show that $P_e(2,1,1,0) \leq P_e(4,0,0,0)$. Using equations (41) and (57) we can write:

$$4P_e(4,0,0,0)-4P_e(2,1,1,0) = 3P_F^4+12P_F^3(1-P_F)+6P_D^2(1-P_D)^2+12P_F^2(1-P_F)^2+4P_D(1-P_D)^3$$

$$+ 8P_F(1-P_F)^3 + (1-P_D)^4 + 2(1-P_F)^4 - 2P_D P_F^3 - P_F^4 - 2P_D P_F^2(1-P_F) - 2P_F^3(1-P_D)$$

$$- 2P_F^3(1-P_F) - 2P_F^2(1-P_D)(1-P_F) - P_F^2(1-P_F)^2 - 2P_D P_F^2(1-P_D) - 2P_D P_F^2(1-P_F)$$

$$-2P_F^3(1-P_F)-4P_D P_F(1-P_D)(1-P_F)-4P_F^2(1-P_D)(1-P_F)-4P_F^2(1-P_F)^2-4P_F(1-P_D)(1-P_F)^2$$

$$-2P_F(1-P_F)^3-P_F^2(1-P_D)^2-P_D P_F(1-P_F)^2-P_F^2(1-P_F)^2-2P_F(1-P_F)(1-P_D)^2-2P_F(1-P_D)(1-P_F)^2$$

$$- 2P_F(1-P_F)^3 - (1-P_D)^2(1-P_F)^2 - 2(1-P_D)(1-P_F)^3$$

Simplifying this expression we get:

$$4P_e(4,0,0,0) - 4P_e(2,1,1,0) = (P_D - P_F)\{4 - 5P_F - (P_D + P_F)\} + 2(P_D^2 - P_F^2)(P_D^2 + P_F^2) + P_D^4$$

$$+ P_D P_F^3 + 2P_D^2 P_F + 2P_D^2 P_F(1 - P_F)$$

Using the fact $P_D(1 - P_D) \geq P_F(1 - P_F)$ we get:

$$4P_e(4,0,0,0) \geq 4P_e(2,1,1,0)$$

$$\Rightarrow \ P_e(2,1,1,0) \leq \min\{P_e(2,2,0,0), P_e(3,1,0,0), P_e(4,0,0,0)\}$$

Case CY2: $P_D^3(1-P_D) \geq P_F^3(1-P_F), P_D^2(1-P_D)^2 < P_F^2(1-P_F)^2, P_D(1-P_D)^3 < P_F(1-P_F)^3, P_D^2(1-P_D) \geq P_F^2(1-P_F), P_D(1-P_D)^2 < P_F(1-P_F)^2, P_D(1-P_D) < P_F(1-P_F)$

For this case the probability of error for the placements are given by equations (42), (46), (51), and (57) respectively. We will show that $P_e(2,1,1,0) \leq P_e(2,2,0,0), P_e(3,1,0,0), P_e(4,0,0,0)$ respectively. Using equations (42) and (57) we can write:

$$4P_e(4,0,0,0) - 4P_e(2,1,1,0) = 3P_F^4 + 12P_F^3(1-P_F) + 6P_D^2(1-P_D)^2 + 12P_F^2(1-P_F)^2 + 4P_D(1-P_D)^3$$

$$+ 8P_F(1-P_F)^3 + (1-P_D)^4 + 2(1-P_F)^4 - 2P_D P_F^3 - P_F^4 - 2P_D P_F^2(1-P_F) - 2P_F^3(1-P_D)$$

$$- 2P_F^3(1-P_F) - 2P_F^2(1-P_D)(1-P_F) - P_F^2(1-P_F)^2 - 2P_D P_F^2(1-P_D) - 2P_D P_F^2(1-P_F) - 2P_F^3(1-P_F)$$

$$- 4P_D P_F(1-P_D)(1-P_F) - 4P_F^2(1-P_D)(1-P_F) - 4P_F^2(1-P_F)^2 - 2P_D(1-P_D)(1-P_F)^2$$

$$- 4P_F(1-P_D)(1-P_F)^2 - P_F^2(1-P_D)^2 - P_D P_F(1-P_F)^2 - P_F^2(1-P_F)^2 - 2P_F(1-P_F)(1-P_D)^2$$

$$- 2P_F(1-P_D)(1-P_F)^2 - 2P_F(1-P_F)^3 - (1-P_D)^2(1-P_F)^2 - 2(1-P_D)(1-P_F)^3$$

Simplifying this expression we get:

$$4P_e(4,0,0,0) - 4P_e(2,1,1,0) = (P_D - P_F)\{2 + P_D + 3(P_D + P_F)(P_D^2 + P_F^2) + P_F^3 - 2P_F^2 -$$

$$4P_D^2 - 4P_F^2 - 4P_D P_F\} \geq 0$$

$$\Rightarrow P_e(4,0,0,0) \geq P_e(2,1,1,0)$$

Using equations (42) and (51) we can write:

$$4P_e(3,1,0,0) - 4P_e(2,1,1,0) = P_F^4 + 3P_D^2 P_F(1-P_D) + 4P_F^3(1-P_F) + 6P_F^2(1-P_F)^2 + 3P_D P_F(1-P_D)^2$$

$$+ 3P_D(1-P_F)(1-P_D)^2 + 3P_F(1-P_F)^3 + P_F(1-P_D)^3 + (1-P_D)^3(1-P_F) + (1-P_F)^4 - P_D P_F^3$$

$$- 2P_D P_F^2(1-P_F) - P_F^3(1-P_D) - 2P_D P_F^2(1-P_D) - 2P_D P_F^2(1-P_F) - 4P_D P_F(1-P_D)(1-P_F)$$

$$- 3P_F^2(1-P_D)(1-P_F) - 2P_D(1-P_D)(1-P_F)^2 - 3P_F(1-P_D)(1-P_F)^2 - P_F^2(1-P_D)^2 - P_D P_F(1-P_F)^2$$

$$-2P_F(1-P_F)(1-P_D)^2 - (1-P_D)^2(1-P_F)^2 - (1-P_D)(1-P_F)^3$$

This expression can be simplified to give:

$$4P_e(3,1,0,0) - 4P_e(2,1,1,0) = (P_D - P_F)\{1 - 2(P_D + P_F)(1-P_D)(1-P_F)\} + P_D^2 P_F(1-P_D) +$$

$$P_F^2(1-P_F) + P_F^4 - P_D P_F \geq 0$$

$$\Rightarrow P_e(3,1,0,0) \geq P_e(2,1,1,0)$$

Using equations (42) and (46) we can write:

$$P_e(2,2,0,0) - P_e(2,1,1,0) = P_D^2 P_F^2 + 2P_F^4 + 4P_D P_F^2(1-P_D) + 8P_F^3(1-P_F) + 2P_F^2(1-P_D)^2$$

$$+4P_F^2(1-P_F)^2 + 8P_D P_F(1-P_D)(1-P_F) + 4P_F^2(1-P_F)^2 + 4P_F(1-P_F)(1-P_D)^2 + 4P_D(1-P_D)(1-P_F)^2$$

$$+ 4P_F(1-P_F)^3 + 2(1-P_D)^2(1-P_F)^2 + (1-P_F)^4 - 2P_D P_F^3 - P_F^4 - 2P_D P_F^2(1-P_F)$$

$$- 2P_F^3(1-P_D) - 2P_F^3(1-P_F) - 2P_F^2(1-P_D)(1-P_F) - P_F^2(1-P_F)^2 - 2P_D P_F^2(1-P_D)$$

$$-2P_D P_F^2(1-P_F) - 2P_F^3(1-P_F) - 4P_D P_F(1-P_D)(1-P_F) - 4P_F^2(1-P_D)(1-P_F) - 4P_F^2(1-P_F)^2$$

$$-2P_D(1-P_D)(1-P_F)^2 - 4P_F(1-P_D)(1-P_F)^2 - P_F^2(1-P_D)^2 - P_D P_F(1-P_F)^2 - P_F^2(1-P_F)^2$$

$$-2P_F(1-P_F)(1-P_D)^2 - 2P_F(1-P_D)(1-P_F)^2 - 2P_F(1-P_F)^3 - (1-P_D)^2(1-P_F)^2 - 2(1-P_D)(1-P_F)^3$$

This expression can be simplified to get:

$$4P_e(2,2,0,0) - 4P_e(2,1,1,0) = (P_D - P_F)(1 - P_F^2)\{2(1-P_F) - P_D\} \geq 0$$

$$\Rightarrow P_e(2,2,0,0) \geq P_e(2,1,1,0)$$

From these results we can conclude that:

$$P_e(2,1,1,0) \leq \min\{P_e(2,2,0,0), P_e(3,1,0,0), P_e(4,0,0,0)\}$$

Case CZ2: $P_D^3(1-P_D) \geq P_F^3(1-P_F), P_D^2(1-P_D)^2 < P_F^2(1-P_F)^2, P_D(1-P_D)^3 < P_F(1-P_F)^3, P_D^2(1-P_D) < P_F^2(1-P_F), P_D(1-P_D)^2 < P_F(1-P_F)^2, P_D(1-P_D) < P_F(1-P_F)$

For this case the probability of error for the placements are given by equations (42), (46), (52), and (57) respectively. From CY2 we already know that $P_e(2,1,1,0) \leq 4P_e(4,0,0,0)$. Also using CY2 we can write:

$$4P_e(2,2,0,0) - 4P_e(2,1,1,0) = (P_D - P_F)(1 - P_F^2)\{2(1 - P_F) - P_D\}$$

For $2(1 - P_F) \geq P_D$:

$$P_e(2,1,1,0) \leq P_e(2,2,0,0)$$

For $2(1 - P_F) < P_D$:

$$P_e(2,1,1,0) \geq P_e(2,2,0,0)$$

Using equations (52) and (57) we can write:

$$4P_e(3,1,0,0) - 4P_e(2,2,0,0) = P_D P_F^3 + 2P_F^4 + P_F^3(1 - P_D) + 2P_F^3(1 - P_F) + 3P_D^2 P_F(1 - P_D)$$

$$+ 6P_F^3(1 - P_F) + 3P_D^2(1 - P_D)(1 - P_F) + 3P_F^2(1 - P_D)(1 - P_F) + 3P_F^2(1 - P_F)^2 + 3P_D P_F(1 - P_D)^2$$

$$+ 6P_F^2(1 - P_F)^2 + 3P_D(1 - P_F)(1 - P_D)^2 + 3P_F(1 - P_D)(1 - P_F)^2 + 3P_F(1 - P_F)^3 + P_F(1 - P_D)^3$$

$$+ 2P_F(1 - P_F)^3 + (1 - P_D)^3(1 - P_F) + (1 - P_D)(1 - P_F)^3 + (1 - P_F)^4 - P_D^2 P_F^2 - 2P_F^4$$

$$- 4P_D P_F^2(1 - P_D) - 8P_F^3(1 - P_F) - 2P_F^2(1 - P_D)^2 - 4P_F^2(1 - P_F)^2 - 8P_D P_F(1 - P_D)(1 - P_F)$$

$$- 4P_F^2(1 - P_F)^2 - 4P_F(1 - P_F)(1 - P_D)^2 - 4P_D(1 - P_D)(1 - P_F)^2 - 4P_F(1 - P_F)^3$$

$$- 2(1 - P_D)^2(1 - P_F)^2 - (1 - P_F)^4$$

This expression can be simplified to get:

$$4P_e(3,1,0,0) - 4P_e(2,2,0,0) = 2(P_D^2 - P_F^2) - (P_D - P_F) - (P_D^3 - P_F^3) - P_D P_F^2(P_D - P_F)$$

It should be noted that:

$$2(P_D^2 - P_F^2) - (P_D - P_F) - (P_D^3 - P_F^3) - P_D P_F^2(P_D - P_F) \geq 0$$

$$\Leftrightarrow (P_D + P_F - 1)^2 \leq P_D P_F(1 - P_F)$$

For $(P_D + P_F - 1)^2 \leq P_D P_F(1 - P_F)$:

$$P_e(3, 1, 0, 0) \geq P_e(2, 2, 0, 0)$$

For $(P_D + P_F - 1)^2 > P_D P_F(1 - P_F)$:

$$P_e(3, 1, 0, 0) < P_e(2, 2, 0, 0)$$

Using equations (52) and (42) we can write:

$$4P_e(3, 1, 0, 0) - 4P_e(2, 1, 1, 0) = P_D P_F^3 + 2P_F^4 + P_F^3(1 - P_D) + 3P_D^2 P_F(1 - P_D)$$

$$+6P_F^3(1-P_F)+3P_D^2(1-P_D)(1-P_F)+3P_F^2(1-P_D)(1-P_F)+3P_F^2(1-P_F)^2+3P_D P_F(1-P_D)^2$$

$$+6P_F^2(1-P_F)^2+3P_D(1-P_F)(1-P_D)^2+3P_F(1-P_D)(1-P_F)^2+3P_F(1-P_F)^3+P_F(1-P_D)^3$$

$$+2P_F(1-P_F)^3+(1-P_D)^3(1-P_F)+(1-P_D)(1-P_F)^3+(1-P_F)^4-2P_D P_F^3-P_F^4$$

$$-2P_D P_F^2(1-P_D)-2P_D P_F^2(1-P_F)-2P_F^3(1-P_F)-2P_F^2(1-P_D)(1-P_F)-P_F^2(1-P_F)^2$$

$$-2P_D P_F^2(1-P_D)-2P_D P_F^2(1-P_F)-2P_F^3(1-P_F)-4P_D P_F(1-P_D)(1-P_F)-4P_F^2(1-P_D)(1-P_F)$$

$$-4P_F^2(1-P_F)^2-2P_D(1-P_D)(1-P_F)^2-4P_F(1-P_D)(1-P_F)^2-P_F^2(1-P_D)^2-P_D P_F(1-P_F)^2$$

$$-P_F^2(1-P_F)^2-2P_F(1-P_F)(1-P_D)^2-2P_F(1-P_D)(1-P_F)^2-2P_F(1-P_F)^3$$

$$-(1-P_D)^2(1-P_F)^2-2(1-P_D)(1-P_F)^3$$

Simplifying this expression we get:

$$4P_e(3, 1, 0, 0) - 4P_e(2, 1, 1, 0) = (P_D - P_F)\left\{1 + P_D(1 - P_D) - 2P_F^2(1 - P_F) - P_D P_F - P_F^2\right\}$$

It should be noted that for this case $P_D > P_F$. Using the above expression we get:

For $\{1 + P_D(1 - P_D) - 2P_F^2(1 - P_F) - P_D P_F - P_F^2\} \geq 0$

$$P_e(3, 1, 0, 0) \geq P_e(2, 1, 1, 0)$$

For $\{1 + P_D(1 - P_D) - 2P_F^2(1 - P_F) - P_D P_F - P_F^2\} < 0$

$$P_e(3, 1, 0, 0) < P_e(2, 1, 1, 0)$$

From these comparison equations we obtain 3 distinct subcases which are provided below:

Case CZ2.1: $2(1 - P_F) \geq P_D, (P_D + P_F - 1)^2 \leq P_D P_F(1 - P_F)$

$$P_e(2, 1, 1, 0) \leq P_e(2, 2, 0, 0), P_e(3, 1, 0, 0)$$

Case CZ2.2: $2(1 - P_F) < P_D, (P_D + P_F - 1)^2 \leq P_D P_F(1 - P_F)$

$$P_e(2, 2, 0, 0) \leq P_e(2, 1, 1, 0), P_e(3, 1, 0, 0)$$

Case CZ2.3: $2(1 - P_F) < P_D, (P_D + P_F - 1)^2 > P_D P_F(1 - P_F)$

$$P_e(3, 1, 0, 0) \leq P_e(2, 1, 1, 0), P_e(2, 2, 0, 0)$$

Case DZ2: $P_D^3(1-P_D) < P_F^3(1-P_F), P_D^2(1-P_D)^2 < P_F^2(1-P_F)^2, P_D(1-P_D)^3 < P_F(1-P_F)^3, P_D^2(1-P_D) < P_F^2(1-P_F), P_D(1-P_D)^2 < P_F(1-P_F)^2, P_D(1-P_D) < P_F(1-P_F)$

For this case the probability of error for the placements are given by equations (42), (46), (52), and (58) respectively. Using equations (58) and (52) we can write:

$$4P_e(4,0,0,0)-4P_e(3,1,0,0) = 3P_F^4+4P_D^3(1-P_D)+8P_F^3(1-P_F)+6P_D^2(1-P_D)^2+12P_F^2(1-P_F)^2$$

$$+4P_D(1-P_D)^3+8P_F(1-P_F)^3+(1-P_D)^4+2(1-P_F)^4-P_DP_F^3-2P_F^4-P_F^3(1-P_D)$$

$$-2P_F^3(1-P_F)-3P_D^2P_F(1-P_D)-6P_F^3(1-P_F)-3P_D^2(1-P_D)(1-P_F)-3P_F^2(1-P_D)(1-P_F)$$

$$-3P_F^2(1-P_F)^2-3P_DP_F(1-P_D)^2-6P_F^2(1-P_F)^2-3P_D(1-P_F)(1-P_D)^2-3P_F(1-P_D)(1-P_F)^2$$

$$-3P_F(1-P_F)^3-P_F(1-P_D)^3-2P_F(1-P_F)^3-(1-P_D)^3(1-P_F)-(1-P_D)(1-P_F)^3-(1-P_F)^4$$

Simplifying this expression we get:

$$4P_e(4,0,0,0)-4P_e(3,1,0,0) = -P_D^4-P_F^3+2P_F^4+P_D-P_F+P_D^3-P_DP_F^3$$

For $\{-P_D^4-P_F^3+2P_F^4+P_D-P_F+P_D^3-P_DP_F^3\} \geq 0$

$$P_e(4,0,0,0) \geq P_e(3,1,0,0)$$

For $\{-P_D^4-P_F^3+2P_F^4+P_D-P_F+P_D^3-P_DP_F^3\} < 0$

$$P_e(4,0,0,0) < P_e(3,1,0,0)$$

Using equations (58) and (46) we can write:

$$4P_e(4,0,0,0)-4P_e(2,2,0,0) = 3P_F^4+4P_D^3(1-P_D)+8P_F^3(1-P_F)+6P_D^2(1-P_D)^2+12P_F^2(1-P_F)^2$$

$$+4P_D(1-P_D)^3+8P_F(1-P_F)^3+(1-P_D)^4+2(1-P_F)^4-P_D^2P_F^2-2P_F^4-4P_DP_F^2(1-P_D)$$

$$-8P_F^3(1-P_F)-2P_F^2(1-P_D)^2-4P_F^2(1-P_F)^2-8P_DP_F(1-P_D)(1-P_F)-4P_F^2(1-P_F)^2$$

$$-4P_F(1-P_F)(1-P_D)^2-4P_D(1-P_D)(1-P_F)^2-4P_F(1-P_F)^3-2(1-P_D)^2(1-P_F)^2-(1-P_F)^4$$

Simplifying this expression we get:

$$4P_e(4,0,0,0) - 4P_e(2,2,0,0) = (P_D^2 - P_F^2)(2 - P_D^2 - 2P_F^2)$$

For $(P_D^2 - P_F^2)(2 - P_D^2 - 2P_F^2) \geq 0$

$$P_e(4,0,0,0) \geq P_e(2,2,0,0)$$

For $(P_D^2 - P_F^2)(2 - P_D^2 - 2P_F^2) < 0$

$$P_e(4,0,0,0) < P_e(2,2,0,0)$$

Using equations (42) and (58) we can write:

$$4P_e(4,0,0,0) - 4P_e(2,1,1,0) = 3P_F^4 + 4P_D^3(1-P_D) + 8P_F^3(1-P_F) + 6P_D^2(1-P_D)^2$$

$$+ 12P_F^2(1-P_F)^2 + 4P_D(1-P_D)^3 + 8P_F(1-P_F)^3 + (1-P_D)^4 + 2(1-P_F)^4 - 2P_D P_F^3 - P_F^4$$

$$-2P_D P_F^2(1-P_F) - 2P_F^3(1-P_D) - 2P_F^3(1-P_F) - 2P_F^2(1-P_D)(1-P_F) - P_F^2(1-P_F)^2 - 2P_D P_F^2(1-P_D)$$

$$-2P_D P_F^2(1-P_F) - 2P_F^3(1-P_F) - 4P_D P_F(1-P_D)(1-P_F) - 4P_F^2(1-P_D)(1-P_F) - 4P_F^2(1-P_F)^2$$

$$-2P_D(1-P_D)(1-P_F)^2 - 4P_F(1-P_D)(1-P_F)^2 - P_F^2(1-P_D)^2 - P_D P_F(1-P_F)^2 - P_F^2(1-P_F)^2$$

$$-2P_F(1-P_F)(1-P_D)^2 - 2P_F(1-P_D)(1-P_F)^2 - 2P_F(1-P_F)^3 - (1-P_D)^2(1-P_F)^2 - 2(1-P_D)(1-P_F)^3$$

This expression can be simplified to give:

$$4P_e(4,0,0,0) - 4P_e(2,1,1,0) = 2(P_D - P_F) + P_D(P_D - P_F) - P_D(P_D^3 - P_F^3) - 2P_F^2(P_D - P_F)$$

For: $2(P_D - P_F) + P_D(P_D - P_F) - P_D(P_D^3 - P_F^3) - 2P_F^2(P_D - P_F) \geq 0$

$$P_e(4,0,0,0) \geq P_e(2,1,1,0)$$

For: $2(P_D - P_F) + P_D(P_D - P_F) - P_D(P_D^3 - P_F^3) - 2P_F^2(P_D - P_F) < 0$

$$P_e(4,0,0,0) < P_e(2,1,1,0)$$

Using equations (46) and (52) we can write:

$$4P_e(3,1,0,0) - 4P_e(2,2,0,0) = P_D P_F^3 + 2P_F^4 + P_F^3(1-P_D) + 2P_F^3(1-P_F) + 3P_D^2 P_F(1-P_D)$$

$$+6P_F^3(1-P_F) + 3P_D^2(1-P_D)(1-P_F) + 3P_F^2(1-P_D)(1-P_F) + 3P_F^2(1-P_F)^2 + 3P_D P_F(1-P_D)^2$$

$$+6P_F^2(1-P_F)^2 + 3P_D(1-P_F)(1-P_D)^2 + 3P_F(1-P_D)(1-P_F)^2 + 3P_F(1-P_F)^3 + P_F(1-P_D)^3$$

$$+2P_F(1-P_F)^3 + (1-P_D)^3(1-P_F) + (1-P_D)(1-P_F)^3 + (1-P_F)^4 - P_D^2 P_F^2 - 2P_F^4 - 4P_D P_F^2(1-P_D)$$

$$-8P_F^3(1-P_F) - 2P_F^2(1-P_D)^2 - 4P_F^2(1-P_F)^2 - 4P_F(1-P_F)^3 - 8P_D P_F(1-P_D)(1-P_F) - 4P_F^2(1-P_F)^2$$

$$- 4P_F(1-P_F)(1-P_D)^2 - 4P_D(1-P_D)(1-P_F)^2 - 2(1-P_D)^2(1-P_F)^2 - (1-P_F)^4$$

Simplifying this epxression we get:

$$4P_e(3,1,0,0) - 4P_e(2,2,0,0) = 2(P_D^2 - P_F^2) - (P_D - P_F) - (P_D^3 - P_F^3) - P_D P_F^2(P_D - P_F)$$

For: $2(P_D^2 - P_F^2) - (P_D - P_F) - (P_D^3 - P_F^3) - P_D P_F^2(P_D - P_F) \geq 0$

$$P_e(3,1,0,0) \geq P_e(2,2,0,0)$$

For: $2(P_D^2 - P_F^2) - (P_D - P_F) - (P_D^3 - P_F^3) - P_D P_F^2(P_D - P_F) < 0$

$$P_e(3,1,0,0) < P_e(2,2,0,0)$$

Using equations (42) and (52) we can write:

$$4P_e(3,1,0,0) - 4P_e(2,1,1,0) = P_D P_F^3 + 2P_F^4 + P_F^3(1-P_D) + 2P_F^3(1-P_F) + 3P_D^2 P_F(1-P_D) +$$

$$6P_F^3(1-P_F) + 3P_D^2(1-P_D)(1-P_F) + 3P_F^2(1-P_D)(1-P_F) + 3P_F^2(1-P_F)^2 + 3P_D P_F(1-P_D)^2$$

$$+6P_F^2(1-P_F)^2 + 3P_D(1-P_F)(1-P_D)^2 + 3P_F(1-P_D)(1-P_F)^2 + 3P_F(1-P_F)^3 + P_F(1-P_D)^3$$

$$+2P_F(1-P_F)^3 + (1-P_D)^3(1-P_F) + (1-P_D)(1-P_F)^3 + (1-P_F)^4 - 2P_D P_F^3 - P_F^4 - 2P_D P_F^2(1-P_F)$$

$$-2P_F^3(1-P_D) - 2P_F^3(1-P_F) - 2P_F^2(1-P_D)(1-P_F) - P_F^2(1-P_F)^2 - 2P_D P_F^2(1-P_D) - 2P_D P_F^2(1-P_F)$$

$$-2P_F^3(1-P_F)-4P_DP_F(1-P_D)(1-P_F)-4P_F^2(1-P_D)(1-P_F)-4P_F^2(1-P_F)^2-2P_D(1-P_D)(1-P_F)^2$$

$$-4P_F(1-P_D)(1-P_F)^2-P_F^2(1-P_D)^2-P_DP_F(1-P_F)^2-P_F^2(1-P_F)^2-2P_F(1-P_D)(1-P_F)^2$$

$$-2P_F(1-P_F)^3-(1-P_D)^2(1-P_F)^2-2(1-P_D)(1-P_F)^3$$

Simplifying this expression we get:

$$4P_e(3,1,0,0)-4P_e(2,1,1,0)=2P_F^3(P_D-P_F)-(P_D^3-P_F^3)-2P_F^2(P_D-P_F)+(P_D-P_F)+P_D(P_D-P_F)$$

For: $2P_F^3(P_D-P_F)-(P_D^3-P_F^3)-2P_F^2(P_D-P_F)+(P_D-P_F)+P_D(P_D-P_F)\geq 0$

$$P_e(3,1,0,0)\geq P_e(2,1,1,0)$$

For: $2P_F^3(P_D-P_F)-(P_D^3-P_F^3)-2P_F^2(P_D-P_F)+(P_D-P_F)+P_D(P_D-P_F)<0$

$$P_e(3,1,0,0)<P_e(2,1,1,0)$$

Using equations (42) and (46) we can write:

$$4P_e(2,2,0,0)-4P_e(2,1,1,0)=P_D^2P_F^2+2P_F^4+4P_DP_F^2(1-P_D)+8P_F^3(1-P_F)+2P_F^2(1-P_D)^2$$

$$+4P_F^2(1-P_F)^2+8P_DP_F(1-P_D)(1-P_F)+4P_F^2(1-P_F)^2+4P_F(1-P_F)(1-P_D)^2+4P_D(1-P_D)(1-P_F)^2$$

$$+4P_F(1-P_F)^3+2(1-P_D)^2(1-P_F)^2+(1-P_F)^4-2P_DP_F^3-P_F^4-2P_DP_F^2(1-P_F)-2P_F^3(1-P_D)$$

$$-2P_F^3(1-P_F)-2P_F^2(1-P_D)(1-P_F)-P_F^2(1-P_F)^2-2P_DP_F^2(1-P_D)-2P_DP_F^2(1-P_F)-2P_F^3(1-P_F)$$

$$-4P_DP_F(1-P_D)(1-P_F)-4P_F^2(1-P_D)(1-P_F)-4P_F^2(1-P_F)^2-2P_D(1-P_D)(1-P_F)^2-4P_F(1-P_D)(1-P_F)^2$$

$$-P_F^2(1-P_D)^2-P_DP_F(1-P_F)^2-P_F^2(1-P_F)^2-2P_F(1-P_F)(1-P_D)^2-2P_F(1-P_D)(1-P_F)^2-2P_F(1-P_F)^3$$

$$-(1-P_D)^2(1-P_F)^2-2(1-P_D)(1-P_F)^3$$

Simplifying this expression we get:

$$4P_e(2,2,0,0)-4P_e(2,1,1,0)=-2P_F^4-P_D^2+2P_F^2+P_D^2P_F^2-2P_F+P_DP_F^3-2P_DP_F^2$$

$$- P_D P_F + 2P_F^3 + 2P_D = 2(P_D - P_F) + 2P_F^3(1 - P_F) - P_D P_F^2(1 - P_F)$$

$$- P_D P_F^2(1 - P_D) - (P_D^2 - P_F^2) - P_F(P_D - P_F)$$

For: $2(P_D - P_F) + 2P_F^3(1 - P_F) - P_D P_F^2(1 - P_F) - P_D P_F^2(1 - P_D) - (P_D^2 - P_F^2) - P_F(P_D - P_F) \geq 0$

$$P_e(2, 2, 0, 0) \geq P_e(2, 1, 1, 0)$$

For: $2(P_D - P_F) + 2P_F^3(1 - P_F) - P_D P_F^2(1 - P_F) - P_D P_F^2(1 - P_D) - (P_D^2 - P_F^2) - P_F(P_D - P_F) \geq 0$

$$P_e(2, 2, 0, 0) < P_e(2, 1, 1, 0)$$

Using the results from Cases AW1, BW1, BX1, CX1, CY2, CZ2, and DZ2 we obtain the following optimality regions on the (P_F, P_D) plane for $P_D > P_F$:

$(4, 0, 0, 0)$ is the Optimal Placement if:

$$\left[\left\{ -P_D^4 - P_F^3 + 2P_F^4 + P_D - P_F + P_D^3 - P_D P_F^3 < 0 \right\} \cap \left\{ (P_D^2 - P_F^2)(2 - P_D^2 - 2P_F^2) < 0 \right\} \cap \left\{ 2(P_D - P_F) + \right. \right.$$

$$\left. \left. + P_D(P_D - P_F) - P_D(P_D^3 - P_F^3) - 2P_F^2(P_D - P_F) < 0 \right\} \cap \left\{ P_D^3(1 - P_D) < P_F^3(1 - P_F) \right\} \right]$$

$(3, 1, 0, 0)$ is the Optimal Placement if:

$$\left[\left\{ -P_D^4 - P_F^3 + 2P_F^4 + P_D - P_F + P_D^3 - P_D P_F^3 \geq 0 \right\} \cap \left\{ 2(P_D^2 - P_F^2) - (P_D - P_F) - (P_D^3 - P_F^3) \right. \right.$$

$$\left. -P_D P_F^2(P_D - P_F) < 0 \right\} \cap \left\{ 2P_F^3(P_D - P_F) - (P_D^3 - P_F^3) - 2P_F^2(P_D - P_F) + (P_D - P_F) + P_D(P_D - P_F) < 0 \right\}$$

$$\cap \left\{ P_D^3(1 - P_D) < P_F^3(1 - P_F) \right\} \right] \bigcup \left[\left\{ 2(1 - P_F) < P_D \right\} \cap \left\{ (P_D + P_F - 1)^2 > P_D P_F(1 - P_F) \right\} \right.$$

$$\cap \left\{ P_D^3(1 - P_D) \geq P_F^3(1 - P_F) \right\} \cap \left\{ P_D^2(1 - P_D) < P_F^2(1 - P_F) \right\} \right]$$

$(2, 2, 0, 0)$ is the Optimal Placement if:

$$\left[\left\{ (P_D^2 - P_F^2)(2 - P_D^2 - 2P_F^2) \right\} \cap \left\{ 2(P_D^2 - P_F^2) - (P_D - P_F) - (P_D^3 - P_F^3) - P_D P_F^2(P_D - P_F) \geq 0 \right\} \right.$$

$$\left\{ 2(P_D - P_F) + 2P_F^3(1 - P_F) - P_D P_F^2(1 - P_F) - P_D P_F^2(1 - P_D) - (P_D^2 - P_F^2) - P_F(P_D - P_F) < 0 \right\}$$

$$\cap \left\{ P_D^3(1-P_D) < P_F^3(1-P_F) \right\} \Big] \bigcup \Big[\left\{ 2(1-P_F) < P_D \right\} \cap \left\{ (P_D+P_F-1)^2 \le P_D P_F (1-P_F) \right\}$$

$$\cap \left\{ P_D^3(1-P_D) \ge P_F^3(1-P_F) \right\} \cap \left\{ P_D^2(1-P_D) < P_F^2(1-P_F) \right\} \Big]$$

$(2,1,1,0)$ is the Optimal Placement if:

$$\Big[P_D(1-P_D) \ge P_F(1-P_F) \Big] \bigcup \Big[\left\{ P_D(1-P_D) < P_F(1-P_F) \right\} \cap \left\{ P_D^2(1-P_D) \ge P_F^2(1-P_F) \right\} \Big]$$

$$\bigcup \Big[\left\{ 2(1-P_F) \ge P_D \right\} \cap \left\{ (P_D+P_F-1)^2 \le P_D P_F(1-P_F) \right\} \cap \left\{ P_D^3(1-P_D) \ge P_F^3(1-P_F) \right\}$$

$$\left\{ P_D^2(1-P_D) < P_F^2(1-P_F) \right\} \Big] \bigcup \Big[\left\{ 2(P_D-P_F)+P_D(P_D-P_F)-P_D(P_D^3-P_F^3)-2P_F^2(P_D-P_F) \ge 0 \right\}$$

$$\left\{ 2P_F^3(P_D-P_F)-(P_D^3-P_F^3)-2P_F^2(P_D-P_F)+(P_D-P_F)+P_D(P_D-P_F) \ge 0 \right\} \cap \left\{ 2(P_D-P_F)+2P_F^3(1-P_F) \right.$$

$$\left. -P_D P_F^2(1-P_F)-P_D P_F^2(1-P_D)-(P_D^2-P_F^2)-P_F(P_D-P_F) \ge 0 \right\} \cap \left\{ P_D^3(1-P_D) < P_F^3(1-P_F) \right\} \Big]$$

The aforementioned regions can be simplified to get the following more concise description of these regions:

$(4,0,0,0)$ is the optimal placement if:

$$\Big[(P_D - P_F)\big[-(P_D + P_F)(P_D^2 + P_F^2) + (P_D^2 + P_D P_F + P_F^2) + (1 - P_F^3) \big] < 0 \Big] \tag{59}$$

$(3,1,0,0)$ is the optimal placement if:

$$\Big[\left\{ (P_D-P_F)\big[-(P_D+P_F)(P_D^2+P_F^2)+(P_D^2+P_D P_F+P_F^2)+(1-P_F^3) \big] \ge 0 \right\} \cap \left\{ 2(P_D^2-P_F^2)-(P_D-P_F)- \right.$$

$$\left. (P_D^3-P_F^3)-P_D P_F^2(P_D-P_F) < 0 \right\} \cap \left\{ P_D^3(1-P_D) < P_F^3(1-P_F) \right\} \Big] \bigcup \Big[\left\{ (P_D+P_F-1)^2 > P_D P_F(1-P_F) \right\}$$

$$\cap \left\{ P_D^3(1-P_D) \ge P_F^3(1-P_F) \right\} \cap \left\{ P_D^2(1-P_D) < P_F^2(1-P_F) \right\} \Big] \tag{60}$$

$(2,2,0,0)$ is the optimal placement if:

$$\Big[\left\{ (P_D^2-P_F^2)(2-P_D^2-2P_F^2) \ge 0) \right\} \cap \left\{ 2(P_D^2-P_F^2)-(P_D-P_F)-(P_D^3-P_F^3)-P_D P_F^2(P_D-P_F) \ge 0 \right\}$$

$$\cap \left\{ 2(P_D - P_F) + 2P_F^3(1 - P_F) - P_D P_F^2 (1 - P_F) - P_D P_F^2 (1 - P_D) - (P_D^2 - P_F^2) - P_F(P_D - P_F) \leq 0 \right\}$$

$$\cap \left\{ P_D^3(1 - P_D) < P_F^3(1 - P_F) \right\} \right] \bigcup \left[\left\{ 2(1 - P_F) < P_D \right\} \cap \left\{ (P_D + P_F - 1)^2 \leq P_D P_F (1 - P_F) \right\} \right.$$

$$\cap \left\{ P_D^3(1 - P_D) \geq P_F^3(1 - P_F) \right\} \right] \tag{61}$$

$(2, 1, 1, 0)$ is the optimal placement if:

$$\left[2(1 - P_F) \geq P_D \right] \tag{62}$$

For $P_D = P_F$, all the placements have the same probability of error and therefore we choose either the highest placement in the majorization ordering to be the optimal placement on this boundary or select other higher placements which preserve the statement of this proposition. Using such a selection policy at the boundary $P_D = P_F$ and equations (59), (60), (61), and (62) we observe that by fixing P_D in the interval $[0, \frac{2}{3}]$ and by increasing P_F, from 0 to P_D, $(2, 1, 1, 0)$ will be the optimal placement. For P_D fixed in the interval $(\frac{2}{3}, \frac{373}{539}]$ and by increasing P_F from 0 to P_D, the optimal placement will start as the $(2, 1, 1, 0)$ placement and will then switch over to the $(2, 2, 0, 0)$ placement. For P_D fixed in the interval $(\frac{373}{539}, \frac{947}{1093}]$ and by increasing P_F from 0 to P_D, the optimal placement will start as the $(2, 1, 1, 0)$ placement, switch over to the $(2, 2, 0, 0)$ placement, and will end up as the $(3, 1, 0, 0)$ placement. Finally for P_D fixed in the interval $(\frac{947}{1093}, 1]$ and for P_F increased from 0 to P_D, the optimal placement will start as the $(2, 1, 1, 0)$ placement, then switch over to the $(2, 2, 0, 0)$ placement first and then to the $(3, 1, 0, 0)$ placement, and finally end up as the $(4, 0, 0, 0)$ placement. Similarly it can be shown, from equations (59), (60), (61), and (62), that for fixed P_F and increasing P_D the optimal placement occurs nondecreasingly on a majorization based placement scale. We present a plot of (59), (60), (61), and (62) on the (P_F, P_D) plane in Section VI which further validates the statement of this proposition for the case $M = N = 4$. Other cases where $M \leq 5$ can be proved similarly by using equation (5). ∎

For $M > 5$ the aforementioned result does not necessarily hold. Consider the case where $(M, N) = (7, 8)$. Clearly $(3, 2, 1, 1, 0, 0, 0) \succ (2, 2, 2, 1, 0, 0, 0)$, so $(3, 2, 1, 1, 0, 0, 0)$ is at a higher level than $(2, 2, 2, 1, 0, 0, 0)$ on the majorization based placement scale for this problem. Using equation (5) we get $(3, 2, 1, 1, 0, 0, 0)$ as the strictly optimal placement for $(P_F, P_D) =$

$(0.46, 0.6)$ whereas $(2, 2, 2, 1, 0, 0, 0)$ is the strictly optimal placement for $(P_F, P_D) = (0.48, 0.6)$. Therefore, for fixed P_D and increasing P_F the optimal placement does not necessarily occur nondecreasingly on a majorization based placement scale for $M > 5$. Also for $(P_F, P_D) = (0.48, 0.5)$ the placement $(3, 2, 1, 1, 0, 0, 0)$ is strictly optimal whereas $(2, 2, 2, 1, 0, 0, 0)$ is the strictly optimal placement for $(P_F, P_D) = (0.48, 0.6)$. Therefore, we can conclude that for fixed P_F and increasing P_D the optimal placement does not necessarily occur nondecreasingly on a majorization based placement scale for $M > 5$. This counterexample shows that the statement of the proposition is not necessarily valid for $M > 5$.

VI. SIMULATIONS

Example 6.1: M = 3

Let $(M, N) = (3, 3)$, $P_D > P_F$. A (P_F, P_D) plane using a set of optimal placements is given in Fig.1.

The arrows are used to represent the regions in which a specific placement is optimal. In Fig.
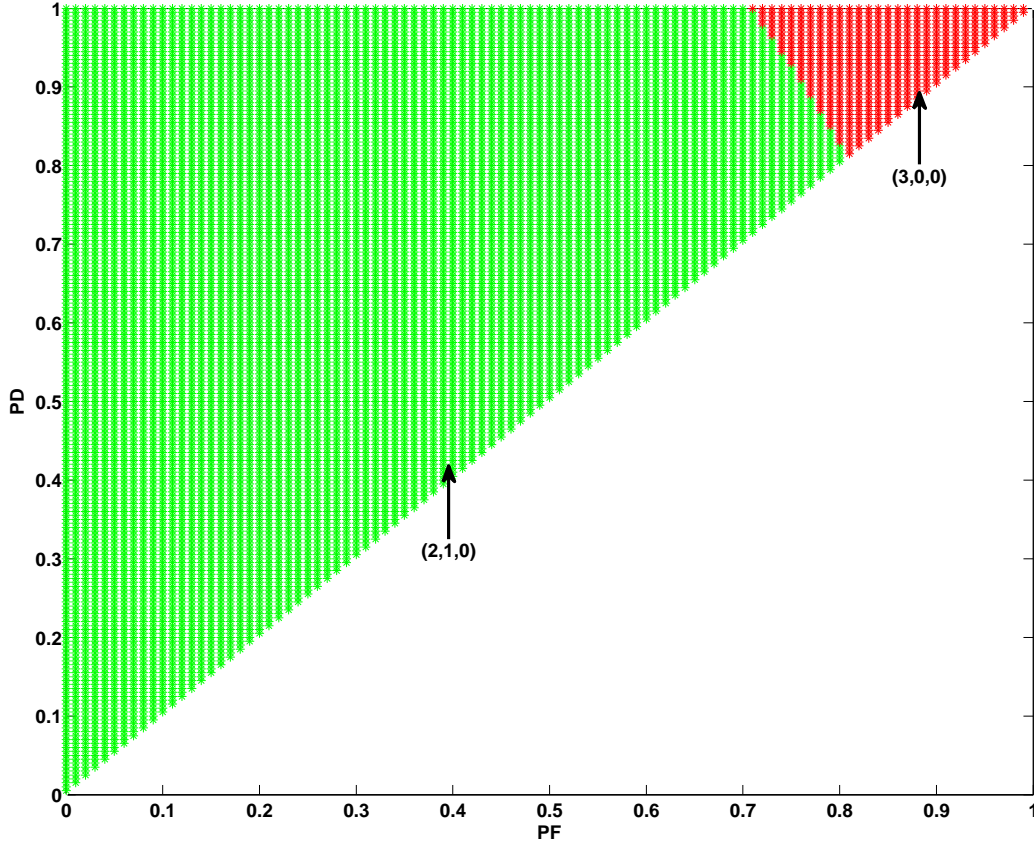

Fig. 1.   Optimal placement structure over (P_F, P_D) for $(M, N) = (3, 3)$

1 we observe that uniform placement is not strictly optimal as specified by Theorem 3.1. By fixing P_D and increasing P_F, we observe that the optimal placement occurs nondecreasingly on the majorization defined scale :$(3, 0, 0) \succ (2, 1, 0) \succ (1, 1, 1)$. Also by fixing P_F and increasing P_D, we observe that the optimal placement occurs nondecreasingly on the majorization defined scale given by (33).

Now consider $(M, N) = (3, 4)$, $P_D > P_F$. A (P_F, P_D) plane using a set of optimal placements is given in Fig. 2. In Fig. 2, we observe that uniform placement is strictly optimal along with
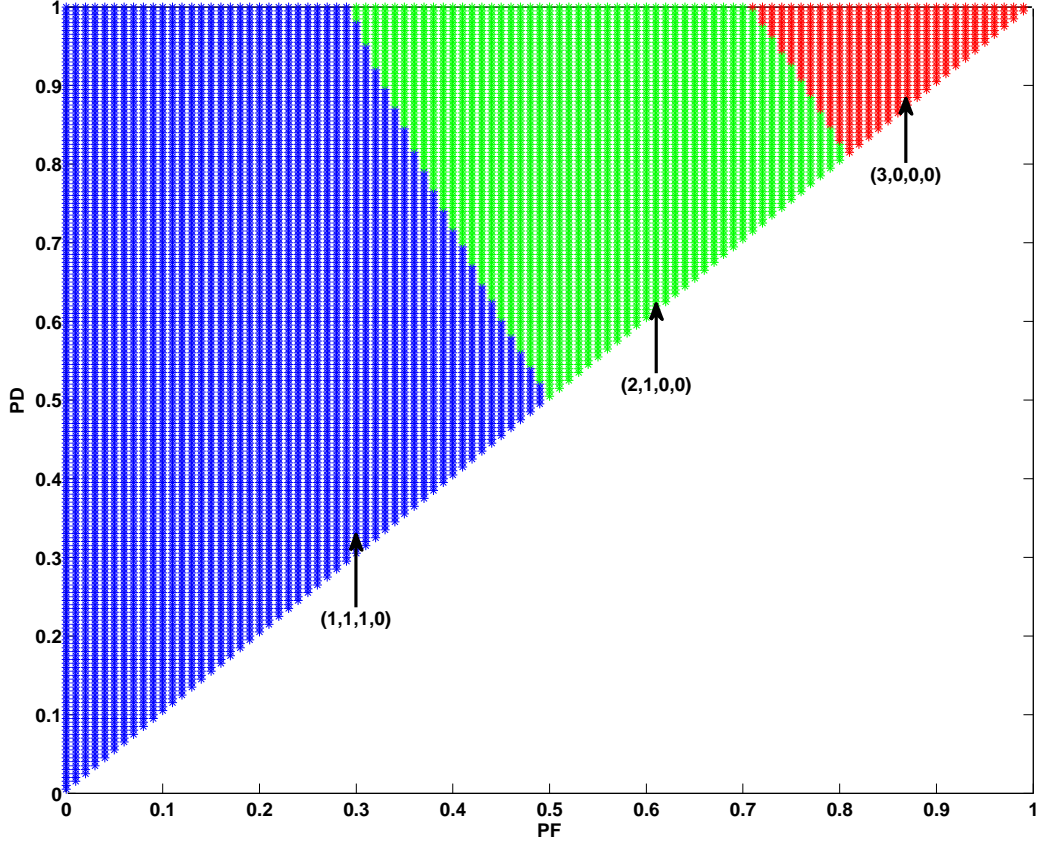
Fig. 2. Optimal placement structure over (P_F, P_D) for $(M, N) = (3, 4)$

the set of optimal placements for the case $(M, N) = (3, 3)$. This was previously prescribed by Corollary 4.1 and these simulations provide a practical validation. As in the case $(M, N) = (3, 3)$ the result of Proposition 5.1 holds for this case and can easily be verified using Fig. 2.

Example 6.2: M = 4

Let $(M, N) = (4, 4)$, $P_D > P_F$. A (P_F, P_D) plane using a set of optimal placements is given in Fig.3. The arrows are used to represent the regions in which a specific placement is optimal.
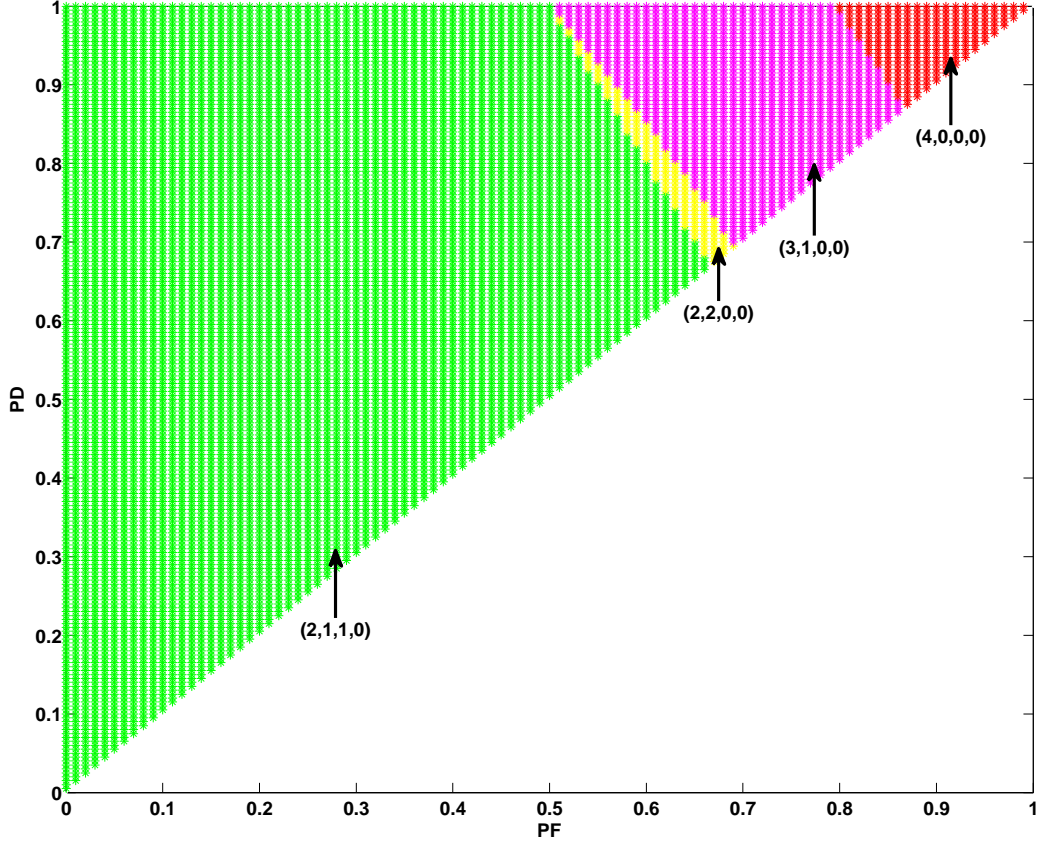

Fig. 3.   Optimal placement structure over (P_F, P_D) for $(M, N) = (4, 4)$

In Fig. 3 we observe that uniform placement is not strictly optimal. By fixing P_D and increasing P_F, we observe that the optimal placement occurs nondecreasingly on the majorization defined scale given by (33). Also by fixing P_F and increasing P_D, we observe that the optimal placement occurs nondecreasingly on the majorization defined scale given by (33).

Now consider $(M, N) = (4, 5)$, $P_D > P_F$. A (P_F, P_D) plane using a set of optimal placements is given in Fig. 4. In Fig. 4, we observe that uniform placement is strictly optimal along with the set of optimal placements for the case $(M, N) = (4, 4)$. This was previously prescribed by Corollary 4.1 and these simulations provide a practical validation. As in the case $(M, N) = (4, 4)$ the result of Proposition 5.1 holds for this case and can easily be verified using Fig. 4.
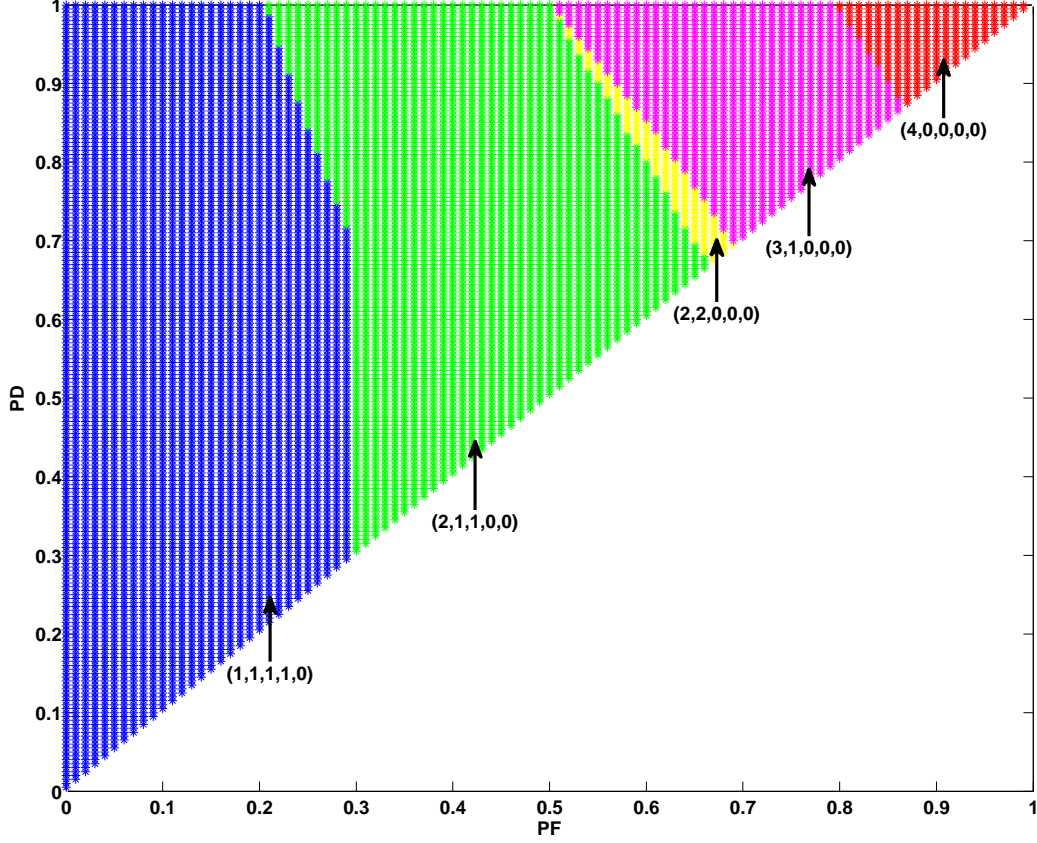
Fig. 4. Optimal placement structure over (P_F, P_D) for $(M, N) = (4, 5)$

Example 6.3: M = 5

Consider the case $(M, N) = (5, 6)$, $P_D > P_F$. A (P_F, P_D) plane using a set of optimal placements for this case is given in Fig. 5. In Fig. 5, by fixing either P_D or P_F and increasing the other parameter, we observe that the optimal placement occurs nondecreasingly on the majorization defined placement scale $(5, 0) \succ (4, 1) \succ (3, 2) \succ (2, 2, 1) \succ (2, 1, 1, 1) \succ (1, 1, 1, 1, 1)$. These simulations verify the statements and signifiy the practical importance of the Theorems as outlined in Section IV and Section V.
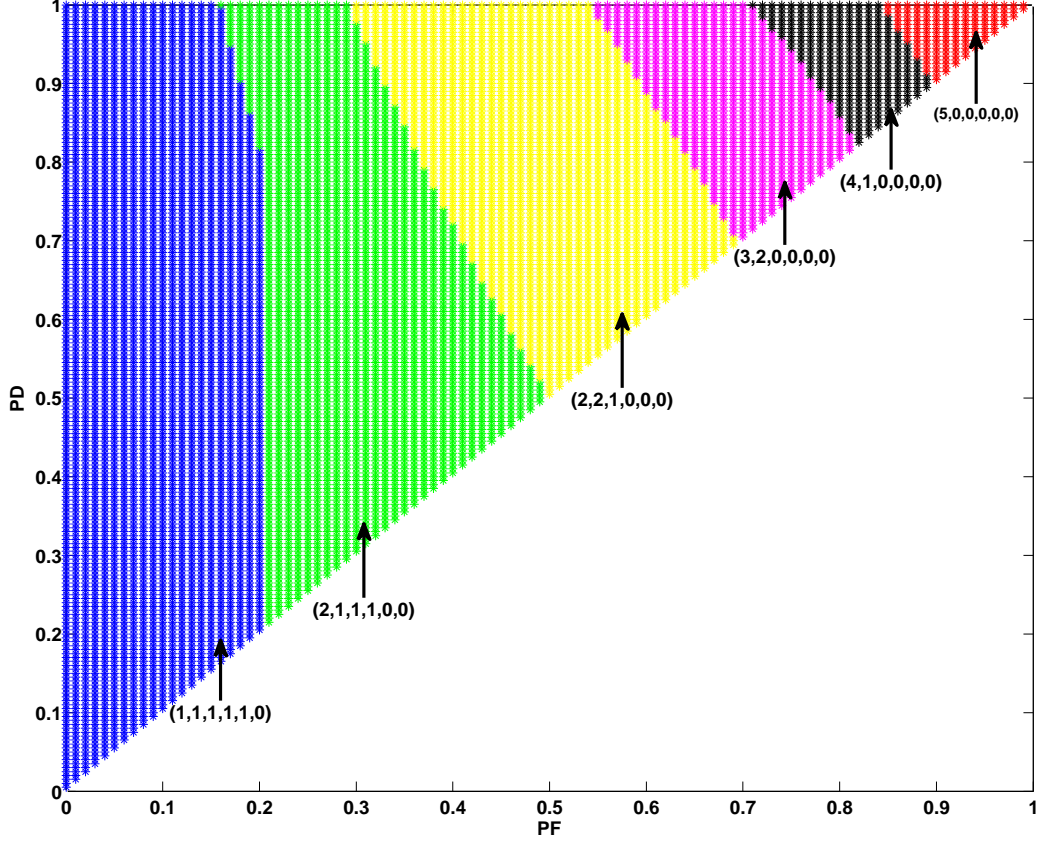
Fig. 5.   Optimal placement structure over (P_F, P_D) for $(M, N) = (5, 6)$

VII. CONCLUSIONS

We have characterized several placement principles for sensors tasked with detecting the location of a randomly placed intruder. In particular, we have used notions from Majorization Theory, such as majorization-based placement scales, to formalize important sensor placement properties. The uniform placement of sensors has been thoroughly analyzed, culminating in an important result on the strict optimality of a uniform placement. In addition, changes in the optimal placement structure due to a variation in the number of points have been analyzed and related design principles have been presented.

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